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The Uniqueness of Phase Retrieval from Intensity Measurements

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I. INTRODUCTION

The question of phase retrieval, that is, of the determination of the phase of an incoming optical signal, is an extremely important problem, one which can be approached in several different ways.

One of the most practical methods of phase retrieval is to attempt to deduce the phase of the wavefield from a knowledge of the intensity of the field in two different planes: the aperture plane and focal plane of a thin lens.

In this report we will investigate several aspects of the uniqueness of the phase retrieval performed by this method, that is, whether or not the method allows one to uniquely determine the phase of the incoming wavefield.

The method of phase retrieval under consideration can be described more precisely in the following way. Let the field in the aperture plane of the lens (which we take to be the plane $z = 0$) be written as

$$\begin{aligned} V(\vec{\rho}_0, t) &= \text{Re} [U_a(\vec{\rho}_0) e^{-i\omega t}] , \\ &= \text{Re} [|U_a(\vec{\rho}_0)| e^{ikW(\vec{\rho}_0)} e^{-i\omega t}] , \end{aligned} \quad (1.1)$$

where $\vec{\rho}_0 = (x_0, y_0)$ is the two-dimensional position vector in the aperture plane, $U_a(\vec{\rho}_0)$ is the complex amplitude of the field in that plane, ω is the temporal frequency of the field and Re denotes that the real part is to be taken. The method in question strives to determine the phase $kW(\vec{\rho}_0)$ from the knowledge of the aperture plane intensity, $|U_a(\vec{\rho}_0)|^2$, and the focal plane intensity, $|U_f(\vec{\rho})|^2$. $\vec{\rho} = (x, y)$ is the position vector in the focal plane. From here on $|U|$ will be

referred to as the amplitude of the field.

Numerical experiments which successfully retrieve the phase from aperture plane and focal plane intensities have been performed. Gon-salves¹ treated the one-dimensional focusing problem. He found, for several test cases, that his algorithm always converged to either the correct phase or its "twin" (the twin solution will be discussed in Section V). Southwell² treated the two-dimensional problem for the case of uniform aperture plane intensity, $|U_a(\vec{\rho}_0)|^2 = 1$, and a phase described by means of the nine Zernike polynomials corresponding to tilt, defocus, astigmatism and coma. He found, for several test cases, that his algorithm produced the correct phase.

Neither of these papers made a definitive statement about the general problem of uniqueness, that is, about whether or not there is only one possible phase function consistent with the two known intensities. Until this point is clarified, one cannot be confident that the phase one calculates is the one occurring physically. In the one-dimensional problem, some interesting results on the question of uniqueness have been obtained.^{3,4,5,6} However, the two-dimensional problem is not a straight-forward extension, because physical wave-functions cannot, in general, be subjected to the method of separation of variables and furthermore lenses have circular apertures, not square ones. Indeed, recently the question of uniqueness for the problem we have described above has been debated in the literature.^{7,8}

More recently two proofs of uniqueness have been obtained. Foley and Butts⁹ have shown that if the wavefield intensity across the lens is uniform and the phase is represented by means of the nine lowest

order Zernike polynomials, that the phase can be determined uniquely (except for the well-known twin solution ambiguity) from its aperture plane and focal plane intensities. Foley¹⁰ has extended this proof to include aperture plane intensities which are Gaussian.

The purpose of this report is to investigate two aspects of the uniqueness question. First, using a specific wavefield (an astigmatic Gaussian beam), we will investigate the role that diffraction plays in the question of uniqueness. (By "diffraction" we mean that due to the finite size of the lens, not diffraction due to unobstructed propagation in free space). It will be shown that this diffraction is responsible for the uniqueness which arises. Secondly, we will discuss the twin solution ambiguity and propose a method for eliminating it.

II. PHASE RETRIEVAL FROM INTENSITY MEASUREMENTS

In this section the particular method of phase retrieval which we are considering, namely the determination of the phase of an incoming complex amplitude wavefield from the knowledge of the intensities in the aperture plane and focal plane of a thin lens, will be described in a more precise mathematical fashion.

2.1 Precise Mathematical Statement of the Uniqueness Problem

Let the wavefield

$$V_1(\vec{r}, t) = \text{Re} \{ U_1(\vec{r}) e^{-i\omega t} \}, \quad (2.1)$$

where $\vec{r} = (x, y, z)$, be traveling towards positive values of z . (From here on the $e^{-i\omega t}$ will be omitted, since the field at any position in space will have this form of time dependence. We will now deal with complex amplitudes only and refer to them as the fields.) Furthermore let this field be incident upon a thin lens of focal length f and radius L , which is located at $z = 0$ and surrounded by a stop. In this plane the incident field can be written in terms of its modulus and phase as

$$U_1(\vec{\rho}_0, 0) = |U_1(\vec{\rho}_0, 0)| e^{ikW(\vec{\rho}_0)}, \quad (2.2)$$

where $\vec{\rho}_0$ = two dimensional position vector in the plane $z = 0$, $k = \omega/c$ and $W(\vec{\rho}_0)$ is a real function of $\vec{\rho}_0$. See Figure 2.1. The aperture plane field (which includes the effect of the stop) is

$$U_a(\vec{\rho}_0) = U_1(\vec{\rho}_0, 0) \text{ circ}(\rho_0/L), \quad (2.3)$$

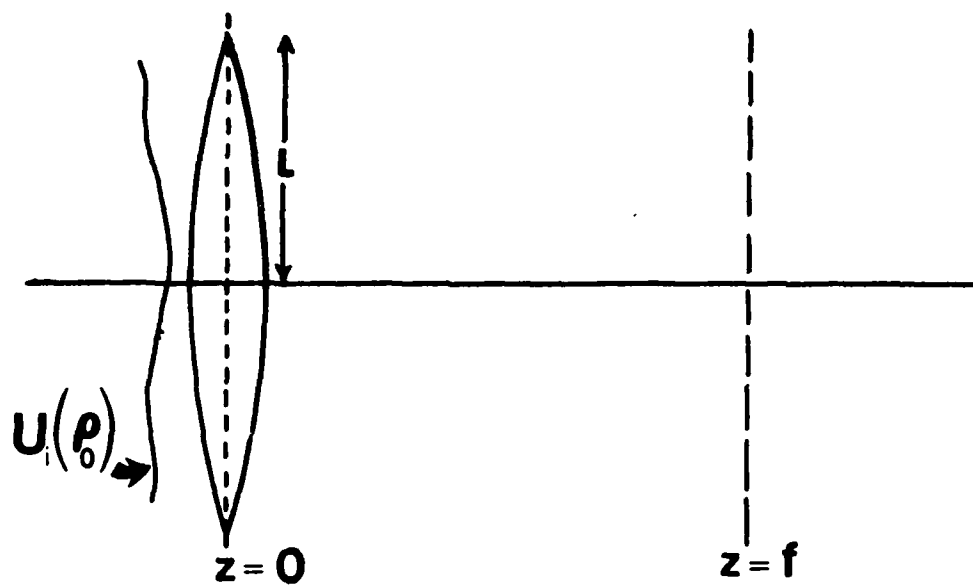


Figure 2.1. Thin Lens Geometry.

where $\rho_0 = |\vec{\rho}_0|$ and

$$\text{circ}(\rho_0/L) = \begin{cases} 1, & \rho_0 \leq L \\ 0, & \rho_0 > L \end{cases}. \quad (2.4)$$

The field, $U_f(\vec{\rho})$, in the focal plane can be found by using diffraction theory. It turns out to be proportional to the Fourier transform of the aperture plane field¹¹,

$$U_f(\vec{\rho}) = \frac{e^{ikf} e^{ik\rho^2/2f}}{i\lambda f} \tilde{U}_a(\vec{\rho}/\lambda f), \quad (2.5)$$

where λ is the wavelength of the light and

$$\tilde{U}_a(\vec{\rho}/\lambda f) = \int_{-\infty}^{\infty} U_a(\vec{\rho}_0) e^{-2\pi i(\vec{\rho}/\lambda f) \cdot \vec{\rho}_0} d^2\rho_0. \quad (2.6)$$

It follows from Eq. (2.5) that the intensity distribution in the focal plane is

$$\begin{aligned} I_f(\vec{\rho}) &= |U_f(\vec{\rho})|^2, \\ &= \frac{1}{\lambda^2 f^2} |\tilde{U}_a(\vec{\rho}/\lambda f)|^2. \end{aligned} \quad (2.7)$$

The phase retrieval problem can be stated as: given the intensity distribution in the focal plane and aperture plane of a lens, can we then uniquely determine the phase of the aperture plane field? Or in other words, is it possible for two (or more) aperture plane fields with equal aperture plane intensities but different phases, e.g.,

$$U_{aA}(\vec{\rho}_0) = |U_a(\vec{\rho}_0)| e^{ikW_A(\vec{\rho}_0)},$$

$$U_{aB}(\vec{\rho}_0) = |U_a(\vec{\rho}_0)| e^{ikW_B(\vec{\rho}_0)},$$

to produce equal focal plane intensities? The equality of the aperture plane intensities demands that

$$|U_{aA}(\vec{\rho}_0)|^2 = |U_{aB}(\vec{\rho}_0)|^2, \text{ for all } \vec{\rho}_0. \quad (2.8)$$

The equality of the focal plane intensities demands that

$$|\tilde{U}_{aA}(\vec{\rho}/\lambda f)|^2 = |\tilde{U}_{aB}(\vec{\rho}/\lambda f)|^2, \text{ for all } \vec{\rho}. \quad (2.9)$$

2.2 The Twin Solution

It is well-known^{1,3,12} that if the aperture plane amplitude is symmetric with respect to inversion, i.e.,

$$|U_a(-\vec{\rho}_0)| = |U_a(\vec{\rho}_0)|, \text{ for all } \vec{\rho}_0, \quad (2.10)$$

then there are at least two fields which will have the same intensities in both the aperture plane and focal plane. They are the field itself,

$$U_{aA}(\vec{\rho}_0) = |U_a(\vec{\rho}_0)| e^{ikW_A(\vec{\rho}_0)} \quad (2.11)$$

and its "twin",

$$U_{aB}(\vec{\rho}_0) = |U_a(\vec{\rho}_0)| e^{-ikW_A(-\vec{\rho}_0)}. \quad (2.12)$$

These two fields have the same aperture plane intensities by definition. Furthermore, as we will shown in Section V,

$$\tilde{U}_{aB}(\vec{\rho}/\lambda f) = [\tilde{U}_{aA}(\vec{\rho}/\lambda f)]^*, \quad (2.13)$$

hence, from Eq. (2.7), we see that they have the same focal plane intensities as well. In Section V we will discuss the twin solution further. In Sections III and IV we must keep in mind that, since we will be dealing with fields which obey Eq. (2.10), the twin solution is always a possibility.

III. THE ROLE OF DIFFRACTION BY LENS APERTURE IN THE QUESTION OF UNIQUENESS. PART I: RADIUS OF BEAM RADIUS OF LENS

Devaney⁸ in his paper "On the uniqueness question in the problem of phase retrieval from intensity measurements" suggested that although the phase retrieval problem that we are discussing was shown by Robinson⁷ to be nonunique in the limit of geometric optics, it does however appear to possess a unique solution (except for the twin solution) within the framework of diffraction theory. Indeed, uniqueness (except for the twin solution) for the problem we are considering has been proven by Foley and Butts⁹.

In this and the next section we will investigate the role of the diffraction caused by the lens aperture in the question of uniqueness. We will study a particular type of incident field, namely an astigmatic Gaussian beam. In this section we will assume that L , the radius of the lens, is much larger than the radius of the incident Gaussian beam, so that there is no diffraction due to the lens aperture. We will find that this leads to a great deal of nonuniqueness, i.e., several beams with different phases can have the same intensity in both the aperture and focal planes. In the next section we will treat the case where the beam radius is not negligible compared to L and we will see that, except for the twin solution, all the nonuniqueness is removed.

3.1 Astigmatic Gaussian Beams

Let us now investigate the case where the incident field is an astigmatic Gaussian beam. The incident field can be written as

$$U_1(\vec{\rho}_0, 0) = A(\vec{\rho}_0) e^{ikW(\vec{\rho}_0)}, \quad (3.1)$$

where

$$A(\vec{\rho}_0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2}, \quad (3.2)$$

is a normalized Gaussian amplitude, and the phase function $W(\vec{\rho}_0)$ can be represented by the expansion,

$$W(\vec{\rho}_0) = \sum_{j=1}^5 a_j Z_j(\vec{\rho}_0/L). \quad (3.3)$$

The a_j 's are expansion coefficients and the Z_j 's are a set of modified Zernike polynomials¹³:

$$Z_1(\vec{\rho}_0/L) = x_0/L, \quad (3.4a)$$

$$Z_2(\vec{\rho}_0/L) = y_0/L, \quad (3.4b)$$

$$Z_3(\vec{\rho}_0/L) = 2(\rho_0/L)^2 - 1, \quad (3.4c)$$

$$Z_4(\vec{\rho}_0/L) = (x_0/L)^2 - (y_0/L)^2, \quad (3.4d)$$

$$Z_5(\vec{\rho}_0/L) = 2(x_0/L)(y_0/L). \quad (3.4e)$$

Each of these terms has a meaningful physical interpretation. The first two terms represent, respectively, the tilt in the x and y directions. The third term corresponds to defocus and the last two terms represent, respectively, 0° and 45° -astigmatism.

Substituting Eq. (3.2), (3.3), and (3.4) into Eq. (3.1), we then have

$$U_1(\vec{\rho}_0, 0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} \exp \left\{ ik \left[\frac{a_1 x_0}{L} + \frac{a_2 y_0}{L} + \frac{a_3 (2\rho_0^2 - 1)}{L^2} + \frac{a_4 (x_0^2 - y_0^2)}{L^2} + \frac{2a_5 x_0 y_0}{L^2} \right] \right\} . \quad (3.5)$$

Furthermore for the sake of mathematical convenience we can drop the linear terms in Eq. (3.5), since their only effect would be to shift the focal plane field distribution by $x = -a_1 f/L$ and $y = -a_2 f/L$.

Eq. (3.5) can then be written as

$$U_1(\vec{\rho}_0, 0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} \exp \{ ik [\alpha_3 (\rho_0^2 - \frac{1}{2} L^2) + \alpha_4 (x_0^2 - y_0^2) + 2\alpha_5 x_0 y_0] \} , \quad (3.6)$$

where

$$\alpha_3 = 2a_3/L^2 , \quad (3.7a)$$

$$\alpha_4 = a_4/L^2 , \quad (3.7b)$$

$$\alpha_5 = a_5/L^2 . \quad (3.7c)$$

Throughout the rest of this report our incident field will be of the form of (3.6). Note that α_3 measures the defocus, α_4 measures the 0° astigmatism and α_5 measures the 45° astigmatism. The tilt terms (a_1 and a_2) are needed only if the focal plane intensity pattern is not centered about $x = y = 0$.

3.2 Propagation of an Astigmatic Gaussian Beam Through a Thin Lens

(Infinite Aperture)

3.2.1 Astigmatic Gaussian Beam with $\alpha_5 = 0$

Let us first consider the case where $\alpha_5 = 0$, since $\alpha_5 \neq 0$ can be reduced to the same form (this will be shown in the next subsection). The field on the left hand side of the lens can then be written as

$$\begin{aligned}
 U_1(\vec{\rho}_0, 0) &= \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} e^{ik\alpha_3(\rho_0^2 - \frac{1}{2}L^2)} e^{ik\alpha_4(x_0^2 - y_0^2)}, \\
 &= \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}ik\alpha_3L^2}}{W_0} e^{-\rho_0^2/W_0^2} e^{ikx_0^2/2R_{x_0}^-} e^{iky_0^2/2R_{y_0}^-}, \quad (3.8)
 \end{aligned}$$

where

$$1/R_{x_0}^- = 2(\alpha_3 + \alpha_4), \quad (3.9a)$$

$$1/R_{y_0}^- = 2(\alpha_3 - \alpha_4). \quad (3.9b)$$

The effect of the lens is to change the wavefront curvature by an amount $-1/f$.¹⁴ Therefore the field on the right hand side of the lens is

$$\begin{aligned}
 U(\vec{\rho}_0, 0^+) &= U_a(\rho_0) e^{-ik\rho_0^2/2f}, \\
 &= \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2}ik\alpha_3L^2}}{W_0} e^{-x_0^2/W_0^2} e^{ikx_0^2/2R_{x_0}^+} \\
 &\quad \times e^{-y_0^2/W_0^2} e^{iky_0^2/2R_{y_0}^+}, \quad (3.10)
 \end{aligned}$$

where

$$1/R_{x_0}^+ = 1/R_{x_0}^- - 1/f, \quad (3.11a)$$

$$1/R_{y0}^+ = 1/R_{y0}^- - 1/f. \quad (3.11b)$$

In Eq. (3.10) we have used the fact that $U_a(\vec{\rho}_0) = U_1(\vec{\rho}_0, 0)$ for the case of an infinite aperture. If we now use the complex radius of curvature notation¹⁵, Eq. (3.18) can be rewritten as

$$U(\vec{\rho}_0, 0^+) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{1}{2} i k \alpha_3 L^2}}{W_0} e^{i k x_0^2 / 2 q_{x0}} e^{i k y_0^2 / 2 q_{y0}}, \quad (3.12)$$

where

$$1/q_{x0} = 1/R_{x0}^+ + i\lambda/\pi W_0^2, \quad (3.13a)$$

$$1/q_{y0} = 1/R_{y0}^+ + i\lambda/\pi W_0^2. \quad (3.13b)$$

3.2.2 Beam propagation ($z > 0$)

We are now ready to calculate the field in any plane $z = \text{constant} > 0$. We will use the Fresnel diffraction integral,¹⁶

$$\begin{aligned} U(\vec{\rho}, z) &= \frac{e^{i k z}}{i \lambda z} \iint_{-\infty}^{\infty} U(\vec{\rho}_0, 0^+) e^{i k (\vec{\rho} - \vec{\rho}_0)^2 / 2 z} d^2 \rho_0, \\ &= \frac{e^{i k z}}{i \lambda z} \iint_{-\infty}^{\infty} U(\vec{\rho}_0, 0^+) e^{i k \rho_0^2 / 2 z} e^{-i k \vec{\rho} \cdot \vec{\rho}_0 / z} d^2 \rho_0. \end{aligned} \quad (3.14)$$

It follows in a straightforward manner,^{15,17} after substituting (3.12) into (3.14), that the resulting field is

$$U(\vec{\rho}, z) = \sqrt{\frac{2}{\pi}} \frac{e^{i \psi(z)}}{W_x(z)} \frac{e^{-x^2/W_x^2(z)} e^{i k x^2 / 2 R_x(z)}}{e^{-y^2/W_y^2(z)} e^{i k y^2 / 2 R_y(z)}} \quad (3.15)$$

where

$$W_x(z) = W_0 \sqrt{(1 + z/R_{x0}^+)^2 + (z/z_R)^2}, \quad (3.16a)$$

$$W_y(z) = W_0 \sqrt{(1 + z/R_{y0}^+)^2 + (z/z_R)^2} , \quad (3.16b)$$

$$R_x(z) = \frac{(1 + z/R_{x0}^+)^2 + (z/z_R)^2}{(1/R_{x0}^+)(1 + z/R_{x0}^+) + (z/z_R^2)} , \quad (3.17a)$$

$$R_y(z) = \frac{(1 + z/R_{y0}^+)^2 + (z/z_R)^2}{(1/R_{y0}^+)(1 + z/R_{y0}^+) + (z/z_R^2)} , \quad (3.17b)$$

$$z_R = \pi W_0^2 / \lambda , \quad (3.18)$$

$$\psi(z) = k(z - \frac{1}{2}\alpha_3 L^2) - \frac{1}{2}\psi_x(z) - \frac{1}{2}\psi_y(z) , \quad (3.19)$$

and

$$\psi_x(z) = \tan^{-1} \left[\frac{\lambda/\pi W_0^2}{(1/R_{x0}^+ + 1/z)} \right] , \quad (3.20a)$$

$$\psi_y(z) = \tan^{-1} \left[\frac{\lambda/\pi W_0^2}{(1/R_{y0}^+ + 1/z)} \right] . \quad (3.20b)$$

The intensity is therefore

$$\begin{aligned} I(\vec{\rho}, z) &= U^*(\vec{\rho}, z) U(\vec{\rho}, z) , \\ &= \frac{2}{\pi} \frac{e^{-2x^2/W_x^2(z)}}{W_x(z)} \frac{e^{-2y^2/W_y^2(z)}}{W_y(z)} . \end{aligned} \quad (3.21)$$

From Eq. (3.21), we see that for any particular z , the isophotes (contours of equal intensities) are specified by

$$x^2/W_x^2(z) + y^2/W_y^2(z) = \text{constant},$$

which are ellipses which can degenerate into circles if $W_x(z) = W_y(z)$. Circular isophotes (if they occur) are observed at only one particular value of $z > 0$. This circular image is called the circle of least confusion.

3.2.3 Propagation of a General Astigmatic Gaussian Beam

Let us now investigate the case where α_5 is not generally zero. (We will be following formalism similar to that developed by Cook.¹⁸) The field on the left side of the lens can then be written as

$$U_1(\vec{\rho}_0, 0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} e^{ikW(\vec{\rho}_0)} \quad (3.22)$$

where

$$W(\vec{\rho}_0) = \alpha_3(\rho_0^2 - \frac{1}{2}L)^2 + \alpha_4(x_0^2 - y_0^2) + 2\alpha_5x_0y_0 \quad (3.23)$$

As in Sec. 3.2.1 we can now write the field on the right hand side of the lens as

$$\begin{aligned} U(\vec{\rho}_0, 0^+) &= U_a(\vec{\rho}_0) e^{-ik\rho_0^2/2f} \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} e^{-ik\rho_0^2/2f} e^{ikW(\vec{\rho}_0)} \end{aligned} \quad (3.24)$$

Using the Fresnel diffraction integral, the field for $z > 0$ is

$$U(\vec{\rho}, z) = \frac{e^{ikz} e^{ik\rho^2/2z}}{i\lambda z} \iint_{-\infty}^{\infty} U(\rho_0, 0^+) e^{ik\rho_0^2/2z} \exp\left[\frac{-ik}{z}(xx_0 + yy_0)\right] dx_0 dy_0 \quad (3.25)$$

Substitution of Eq. (3.24) into (3.25) yields

$$U(\vec{\rho}, z) = \frac{e^{ikz} e^{ik\rho^2/2z}}{i\lambda z} \sqrt{\frac{2}{\pi}} \frac{1}{W_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\rho_0^2/W_0^2} e^{-ik\rho_0^2/2f} e^{ik\rho_0^2/2z} e^{ikW(\vec{\rho}_0)} \\ \times \exp \left[\frac{-ik}{z} (xx_0 + yy_0) \right] dx_0 dy_0 \quad (3.26)$$

The term $e^{2ik\alpha_5 x_0 y_0}$ makes this integral nonseparable and different from the integral we encountered earlier. Thus, we have to go to a rotated coordinate system in which this term vanishes in order to make the integral separable.

Let us now rotate the aperture plane coordinates and the observation plane coordinates counterclockwise by an angle θ about the z -axis. In order to do this we now define a new aperture plane coordinates, (\bar{x}_0, \bar{y}_0) , and a new observation planes coordinates, (\bar{x}, \bar{y}) , according to the transformation equations,

$$\begin{aligned} \bar{x}_0 &= x_0 \cos\theta + y_0 \sin\theta \quad , \\ \bar{y}_0 &= -x_0 \sin\theta + y_0 \cos\theta \quad , \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} \bar{x} &= x \cos\theta + y \sin\theta \quad , \\ \bar{y} &= -x \sin\theta + y \cos\theta \quad . \end{aligned} \quad (3.28)$$

The old coordinates are then given in terms of the new coordinates according to

$$\begin{aligned} x_0 &= \bar{x}_0 \cos\theta - \bar{y}_0 \sin\theta \quad , \\ y_0 &= \bar{x}_0 \sin\theta + \bar{y}_0 \cos\theta \quad , \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} x &= \bar{x} \cos \theta - \bar{y} \sin \theta, \\ y &= \bar{x} \sin \theta + \bar{y} \cos \theta. \end{aligned} \quad (3.30)$$

It is a straightforward matter to show¹⁸ that if θ is chosen according to

$$\cot 2\theta = \alpha_4 / \alpha_5, \quad 0 < 2\theta < \pi, \quad (3.31)$$

then in the rotated frame the term $e^{2ik\alpha_5 \bar{x}_0 \bar{y}_0}$ is eliminated. Indeed, in the rotated frame the field specified by (3.26) transforms to

$$\begin{aligned} \bar{U}(\bar{x}, \bar{y}, z) &= \frac{e^{ikz} e^{ik\bar{\rho}^2/2z}}{i\lambda z} \sqrt{\frac{2}{\pi}} \frac{1}{W_0} \iint_{-\infty}^{\infty} e^{-\bar{\rho}_0^2/W_0^2} e^{-ik\bar{\rho}_0^2/2f} e^{ik\bar{\rho}_0^2/2z} \\ &\quad \times \exp \left\{ ik \left[\alpha_3 (\bar{\rho}_0^2 - \frac{1}{2} L^2) + \bar{\alpha}_4 (\bar{x}_0^2 - \bar{y}_0^2) \right] \right\} \exp \left[\frac{-ik}{z} (\bar{x}\bar{x}_0 + \bar{y}\bar{y}_0) \right] d\bar{x}_0 d\bar{y}_0, \\ &= \sqrt{\frac{2}{\pi}} \frac{e^{ik(z - \frac{1}{2} \alpha_3 L^2)} e^{ik\bar{\rho}^2/2z}}{i\lambda W_0 z} \iint_{-\infty}^{\infty} e^{ik\bar{x}_0^2/2q\bar{x}_0} e^{ik\bar{y}_0^2/2q\bar{y}_0} e^{ik\bar{\rho}_0^2/2z} \\ &\quad \times \exp \left[\frac{-ik}{z} (\bar{x}\bar{x}_0 + \bar{y}\bar{y}_0) \right] d\bar{x}_0 d\bar{y}_0, \end{aligned} \quad (3.32)$$

where

$$\bar{\rho} = \sqrt{\bar{x}^2 + \bar{y}^2}, \quad (3.33a)$$

$$\bar{\rho}_0 = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}, \quad (3.33b)$$

$$\bar{\alpha}_4 = \alpha_4 \cos 2\theta + \alpha_5 \sin 2\theta, \quad (3.34)$$

$$1/q_{\bar{x}_0} = 1/R_{\bar{x}_0}^+ + i\lambda/\pi W_0^2, \quad (3.35a)$$

$$1/q_{\bar{y}0} = 1/R_{\bar{y}0}^+ + i\lambda/\pi W_0^2, \quad (3.35b)$$

$$1/R_{\bar{x}0}^+ = 2(\alpha_3 + \bar{\alpha}_4) - 1/f, \quad (3.36a)$$

$$1/R_{\bar{y}0}^+ = 2(\alpha_3 - \bar{\alpha}_4) - 1/f. \quad (3.36b)$$

Eq. (3.32) has the same form as Eq. (3.14) with Eq. (3.12) substituted into it. It then follows that

$$\bar{U}(\bar{x}, \bar{y}, z) = \sqrt{\frac{2}{\pi}} e^{i\bar{\psi}(z)} \frac{e^{-\bar{x}^2/W_{\bar{x}}^2(z)} e^{ik\bar{x}^2/2R_{\bar{x}}(z)}}{\sqrt{W_{\bar{x}}(z)}} \frac{e^{-\bar{y}^2/W_{\bar{y}}^2(z)} e^{ik\bar{y}^2/2R_{\bar{y}}(z)}}{\sqrt{W_{\bar{y}}(z)}}, \quad (3.37)$$

where

$$W_{\bar{x}}(z) = W_0 \sqrt{(1 + z/R_{\bar{x}0}^+)^2 + (z/z_R)^2}, \quad (3.38a)$$

$$W_{\bar{y}}(z) = W_0 \sqrt{(1 + z/R_{\bar{y}0}^+)^2 + (z/z_R)^2}, \quad (3.38b)$$

$$R_{\bar{x}}(z) = \frac{(1 + z/R_{\bar{x}0}^+)^2 + (z/z_R)^2}{(1 + z/R_{\bar{x}0}^+)/R_{\bar{x}0}^+ + (z/z_R)^2}, \quad (3.39a)$$

$$R_{\bar{y}}(z) = \frac{(1 + z/R_{\bar{y}0}^+)^2 + (z/z_R)^2}{(1 + z/R_{\bar{y}0}^+)/R_{\bar{y}0}^+ + (z/z_R)^2}, \quad (3.39b)$$

$$\bar{\psi}(z) = k(z - \frac{1}{2}\alpha_3 L^2) - \frac{1}{2}\psi_{\bar{x}} - \frac{1}{2}\psi_{\bar{y}}, \quad (3.40)$$

and

$$\psi_{\bar{x}}(z) = \tan^{-1} \left[\frac{\lambda/\pi W_0^2}{1/R_{\bar{x}_0} + 1/z} \right], \quad (3.41a)$$

$$\psi_{\bar{y}}(z) = \tan^{-1} \left[\frac{\lambda/\pi W_0^2}{1/R_{\bar{y}_0} + 1/z} \right]. \quad (3.41b)$$

The intensity in the rotated frame is then given by,

$$\bar{I}(\bar{x}, \bar{y}, z) = \frac{2}{\pi} \frac{e^{-2\bar{x}^2/W_{\bar{x}}^2(z)}}{W_{\bar{x}}(z)} \frac{e^{-2\bar{y}^2/W_{\bar{y}}^2(z)}}{W_{\bar{y}}(z)}. \quad (3.42)$$

It follows then that for any particular z , the isophotes are generally ellipses (Figure 3.1) in a rotated plane. It also follows that circular isophotes (if they occur) are observed at only one particular value of $z > 0$.

3.3 Nonuniqueness in the Case of an Infinite Aperture

The uniqueness question asks: can two (or more) astigmatic beams with equal aperture plane intensities but different phases, e.g.,

$$U_{aA}(\vec{\rho}_0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} e^{ik\alpha_3 \rho_0^2} e^{ik\alpha_4(x_0^2 - y_0^2)} e^{ik\alpha_5 2x_0 y_0},$$

$$U_{aB}(\vec{\rho}_0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} e^{ik\beta_3 \rho_0^2} e^{ik\beta_4(x_0^2 - y_0^2)} e^{ik\beta_5 2x_0 y_0},$$

generate identical focal plane intensities? It follows from Eq. (3.42) that the two fields will have the same focal plane intensities if their x and y beam widths are equal in the focal plane,

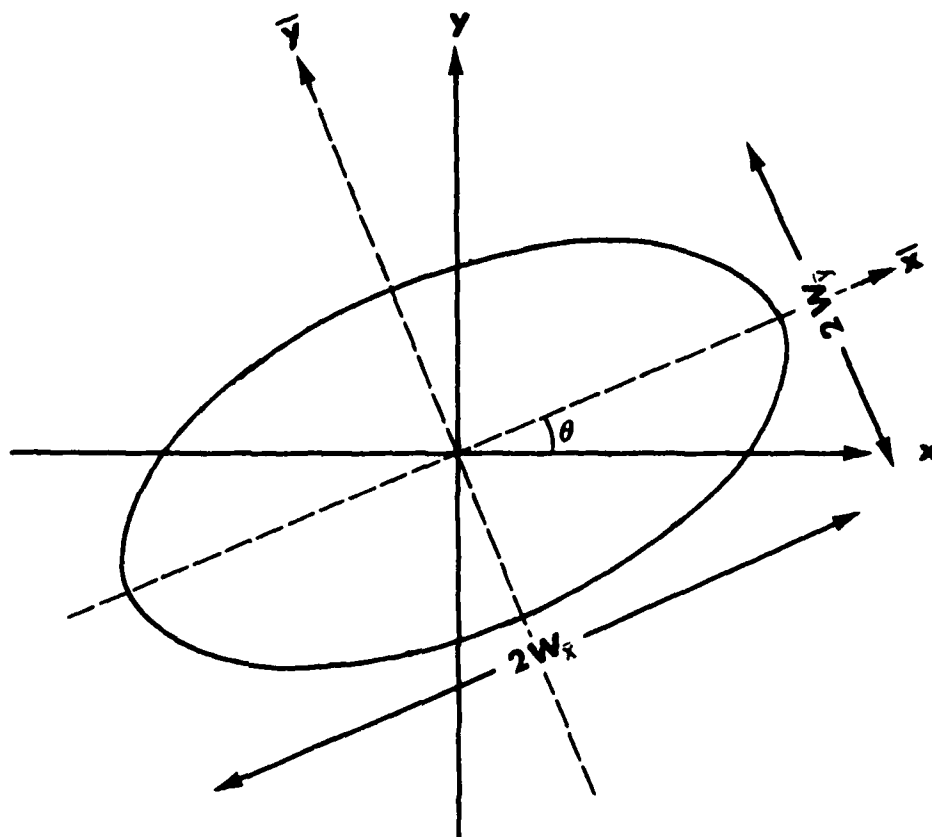


Figure 3.1. Elliptical isophotes for cases with $\alpha_5 \neq 0$.

$$W_{\frac{x}{x}}^A(f) = W_{\frac{x}{x}}^B(f) , \quad (3.43)$$

$$W_{\frac{y}{y}}^A(f) = W_{\frac{y}{y}}^B(f) , \quad (3.44)$$

and the rotation angle of the focal plane isophotes are equal,

$$\theta^A = \theta^B . \quad (3.45)$$

By using Eq. (3.38), (3.36) and (3.34), Eq. (3.43) - (3.45) can be written as

$$\sqrt{4(\alpha_3 + \bar{\alpha}_4)^2 + (1/z_R)^2} = \sqrt{4(\beta_3 + \bar{\beta}_4)^2 + (1/z_R)^2} , \quad (3.46)$$

$$\sqrt{4(\alpha_3 - \bar{\alpha}_4)^2 + (1/z_R)^2} = \sqrt{4(\beta_3 - \bar{\beta}_4)^2 + (1/z_R)^2} , \quad (3.47)$$

$$\frac{\alpha_4}{\alpha_5} = \frac{\beta_4}{\beta_5} . \quad (3.48)$$

Tables, 1, 2, and 3 list all the possible solutions to this set of equations. Table 3.1 gives all the possibilities for the case of $\bar{\alpha}_4 = 0$, Table 3.2 gives all the possibilities for the case of $\alpha_3 = 0$ and Table 3.3 gives all the possibilities for the case where both α_3 and $\bar{\alpha}_4$ are not equal to zero.

It is apparent from Table 3.1 that in this case there are an infinite number of fields which will duplicate the intensity created by the field $\alpha_3 \neq 0$, $\alpha_4 = \alpha_5 = 0$, since we are free to choose β_4 to be anything, as long as $\beta_5 = \pm \sqrt{\alpha_3^2 - \beta_4^2}$. The same is true for some of the solutions in Table 3.2. This occurrence of an infinite number

of solutions is caused by the fact that all the fields created by the solutions in Table 3.1 and Table 3.2 produce circular isophotes (see Appendix A) in the focal plane. Hence, since the aperture plane field is also circularly symmetric any rotation of this incident field (which corresponds to a different β_4 and therefore a different β_5) will leave the focal plane field unchanged as long as β_4 and β_5 are related in the manner stated above. Thus, in practice, if you have an astigmatic beam incident upon a thin lens where $L \gg W_0$, and you see circular isophotes in the aperture plane and focal plane there is no hope of determining the phase of the incident field.

Table 3.3 corresponds to fields which create elliptical isophotes in the focal plane. Here there are only four possible solutions: two distinct solutions, each with its own "twin".

In order to get a physical feel for these four different solutions we set $\alpha_5 = 0$ and look at the propagation of these four beams. The four incident fields are listed in Table 3.4, along with their x and y curvatures. From the table we see that these incident fields all have the same magnitudes for their x curvature and their y curvature; the differences among the solutions is in the signs of the curvatures.

The beam widths in the x direction and y direction, $W_x(z)$ and $W_y(z)$, are plotted as a function of z in Figures 3.2 - 3.5 for z values near the focal plane. The following values were used: $W_0 = \sqrt{2}$ cm., $\lambda = 6328\text{\AA}$, $f = 10$ cm., $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$ and $\alpha_5 = 0$. Note: since $\alpha_5 = 0$, $\theta = 0$ and therefore there is no need to use the rotated coordinate frame. Each of the four beams is quite distinct, except at the point $z = f = 10$ cm. Here the four beams have exactly the same

W_x and W_y and hence, the same intensity ($\theta = 0$ for all four beams). Thus in practice, if you have an astigmatic beam incident upon a thin lens, where $L \gg W_0$, and you see circular isophotes in the aperture plane and elliptical isophotes in the focal plane, you cannot specify which one of the four beams is the one occurring physically.

Indeed, since the second solution is the twin of the first solution, we could not discern between them even if the aperture was smaller (as long as the aperture is circular). Likewise with the third and fourth solutions. If the aperture is small, we can discern between these two groups of solutions, however. This will be shown in the next section.

α_3	α_4	α_5	β_3	β_4	β_5
α_3	0	0	0	β_4	$\sqrt{\alpha_3^2 - \beta_4^2}$
α_3	0	0	0	β_4	$-\sqrt{\alpha_3^2 - \beta_4^2}$
α_3	0	0	α_3	0	0
α_3	0	0	$-\alpha_3$	0	0

Table 3.1 All the possible solutions for the case with $\alpha_4 = 0$.

α_3	α_4	α_5	β_3	β_4	β_5
0	α_4	0	0	β_4	$\sqrt{\alpha_4^2 - \beta_4^2}$
0	α_4	0	0	β_4	$-\sqrt{\alpha_4^2 - \beta_4^2}$
0	α_4	0	α_4	0	0
0	α_4	0	$-\alpha_4$	0	0
0	0	α_5	0	β_4	$\sqrt{\alpha_5^2 - \beta_4^2}$
0	0	α_5	0	β_4	$-\sqrt{\alpha_5^2 - \beta_4^2}$
0	0	α_5	α_5	0	0
0	0	α_5	$-\alpha_5$	0	0
0	α_4	α_5	0	β_4	$\sqrt{\alpha_4^2 + \alpha_5^2 - \beta_4^2}$
0	α_4	α_5	0	β_4	$-\sqrt{\alpha_4^2 + \alpha_5^2 - \beta_4^2}$
0	α_4	α_5	$\sqrt{\alpha_4^2 + \alpha_5^2}$	0	0
0	α_4	α_5	$-\sqrt{\alpha_4^2 + \alpha_5^2}$	0	0

Table 3.2 All possible solutions for $\alpha_3 = 0$.

α_3	α_4	α_5	β_3	β_4	β_5
α_3	α_4	α_5	α_3	α_4	α_5
α_3	α_4	α_5	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$
α_3	α_4	α_5	$\sqrt{\alpha_4^2 + \alpha_5^2}$	$\alpha_3 \alpha_4 / \sqrt{\alpha_4^2 + \alpha_5^2}$	$\alpha_3 \alpha_5 / \sqrt{\alpha_4^2 + \alpha_5^2}$
α_3	α_4	α_5	$-\sqrt{\alpha_4^2 + \alpha_5^2}$	$-\alpha_3 \alpha_4 / \sqrt{\alpha_4^2 + \alpha_5^2}$	$-\alpha_3 \alpha_5 / \sqrt{\alpha_4^2 + \alpha_5^2}$

Table 3.3 All solutions for $\alpha_3 \neq 0$ and $\alpha_4 \neq 0$.

Field in Aperture Plane	X Curvature	Y Curvature
$A(\vec{\rho}_0) e^{ik(\alpha_3 + \alpha_4)x_0^2} e^{ik(\alpha_3 - \alpha_4)y_0^2}$	$\frac{1}{2(\alpha_3 + \alpha_4)}$	$\frac{1}{2(\alpha_3 - \alpha_4)}$
$A(\vec{\rho}_0) e^{-ik(\alpha_3 + \alpha_4)x_0^2} e^{-ik(\alpha_3 - \alpha_4)y_0^2}$	$\frac{-1}{2(\alpha_3 + \alpha_4)}$	$\frac{-1}{2(\alpha_3 - \alpha_4)}$
$A(\vec{\rho}_0) e^{ik(\alpha_3 + \alpha_4)x_0^2} e^{-ik(\alpha_3 - \alpha_4)y_0^2}$	$\frac{1}{2(\alpha_3 + \alpha_4)}$	$\frac{-1}{2(\alpha_3 - \alpha_4)}$
$A(\vec{\rho}_0) e^{-ik(\alpha_3 + \alpha_4)x_0^2} e^{ik(\alpha_3 - \alpha_4)y_0^2}$	$\frac{-1}{2(\alpha_3 + \alpha_4)}$	$\frac{1}{2(\alpha_3 - \alpha_4)}$

Table 3.4. Field and curvature in the x and y direction in the aperture plane for solutions that generate elliptical isophotes in the focal plane.

$$A(\vec{\rho}_0) = \sqrt{\frac{2}{\pi W_0}} e^{-\rho_0^2 / W_0^2}$$

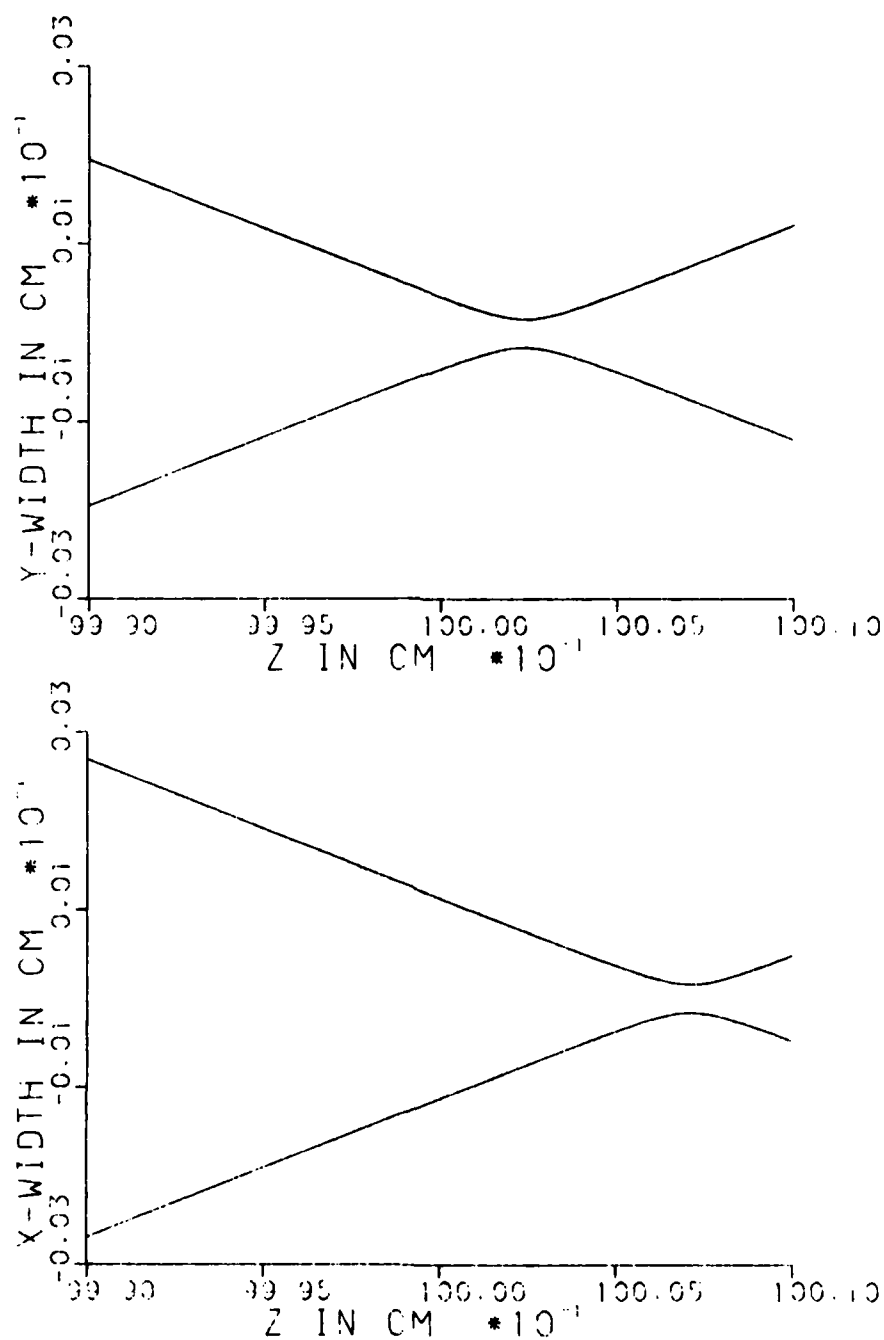


Figure 3.2. Propagation of beam width in the x and y direction for solution 1 of Table 3.3. $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$, $w_0 = \sqrt{2}$ cm, $f = 10$ cm and $\lambda = 6328\text{\AA}$.

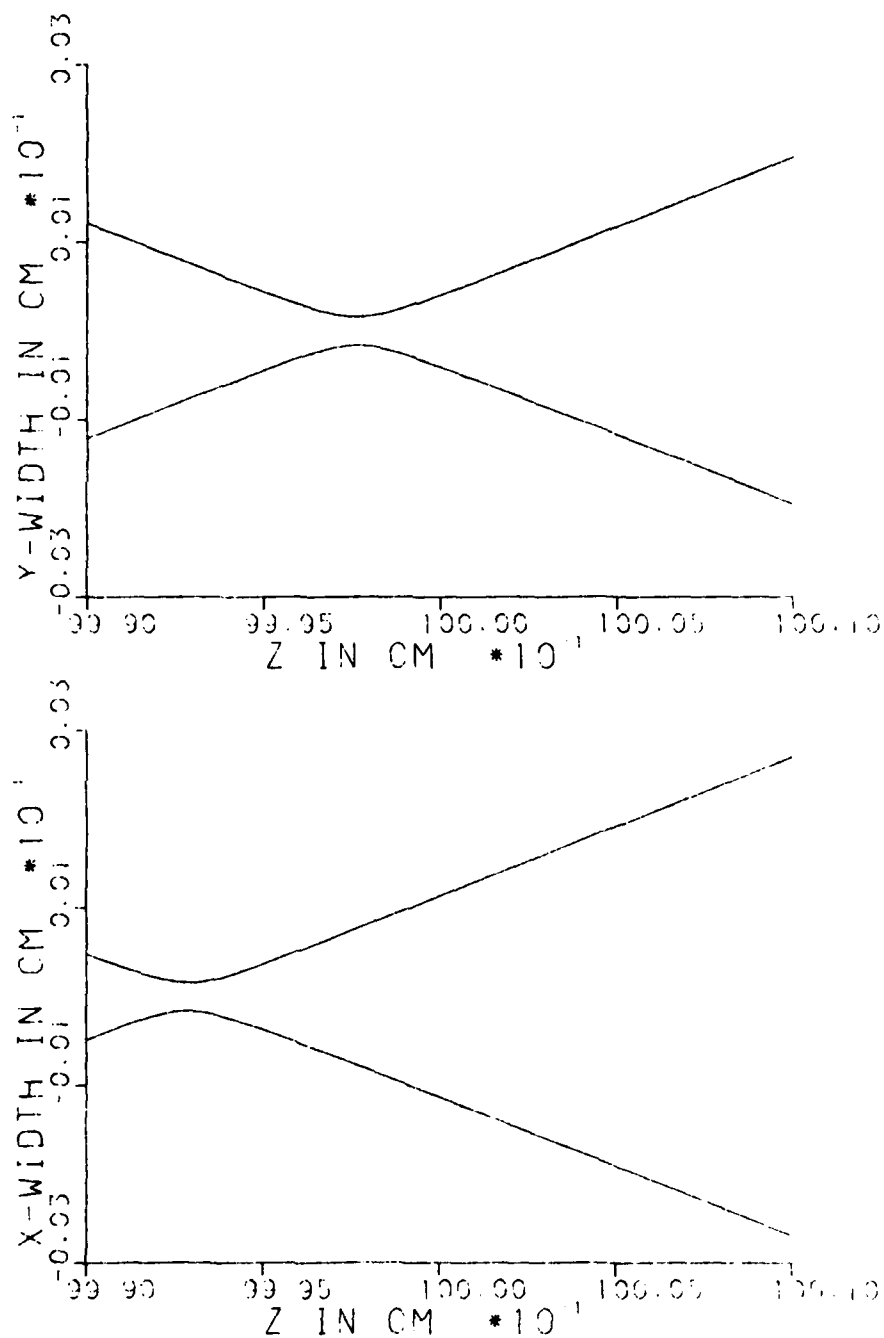


Figure 3.3. Propagation of beam width in the x and y direction for solution 2 of Table 3.3. $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$, $w_0 = \sqrt{2}$ cm, $f = 10$ cm and $\lambda = 6328\text{\AA}$.

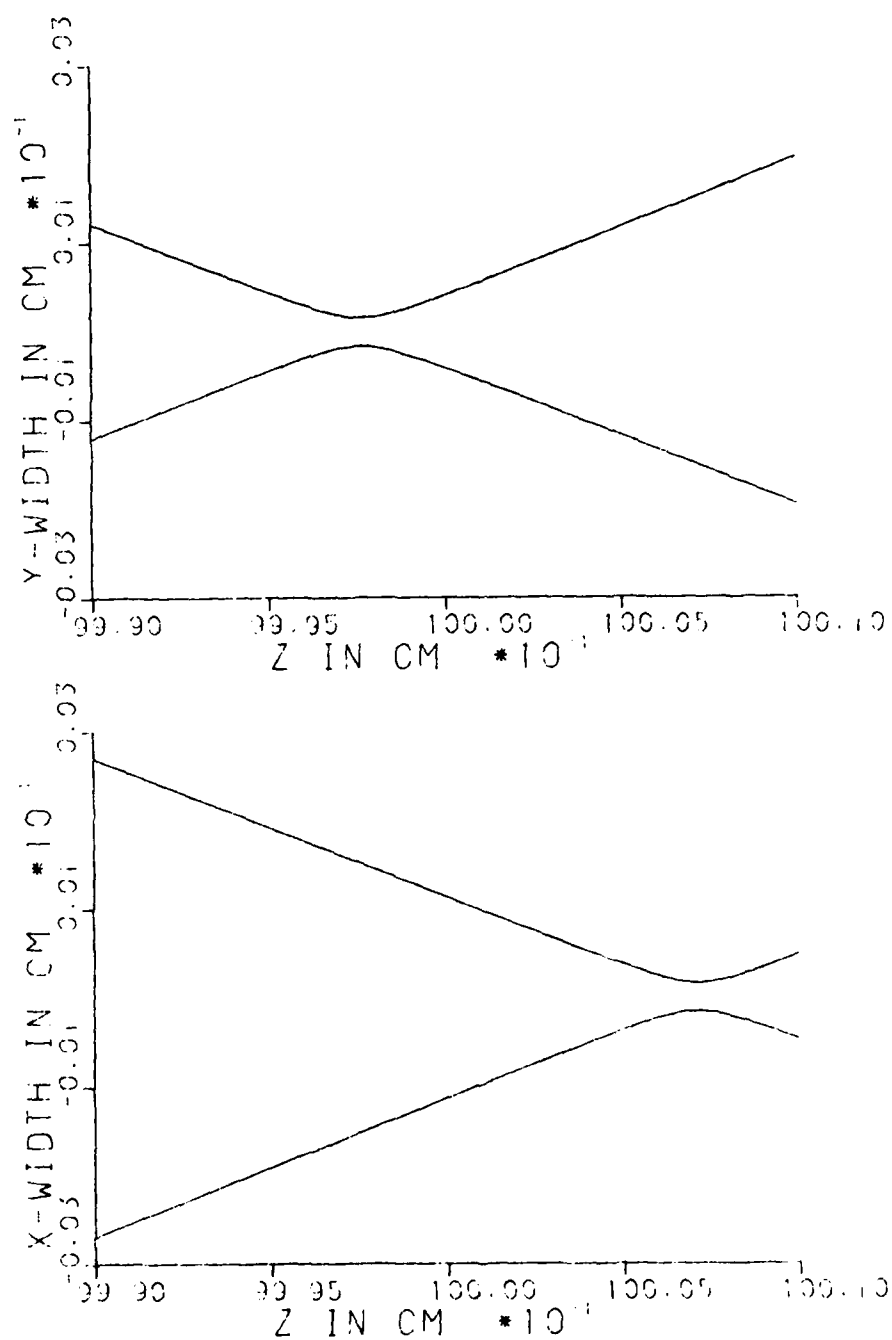


Figure 3.4. Propagation of beam width in the x and y direction for solution 3 of Table 3.3. $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$, $W_0 = \sqrt{2}$ cm, $f = 10$ cm and $\lambda = 6328\text{\AA}$.

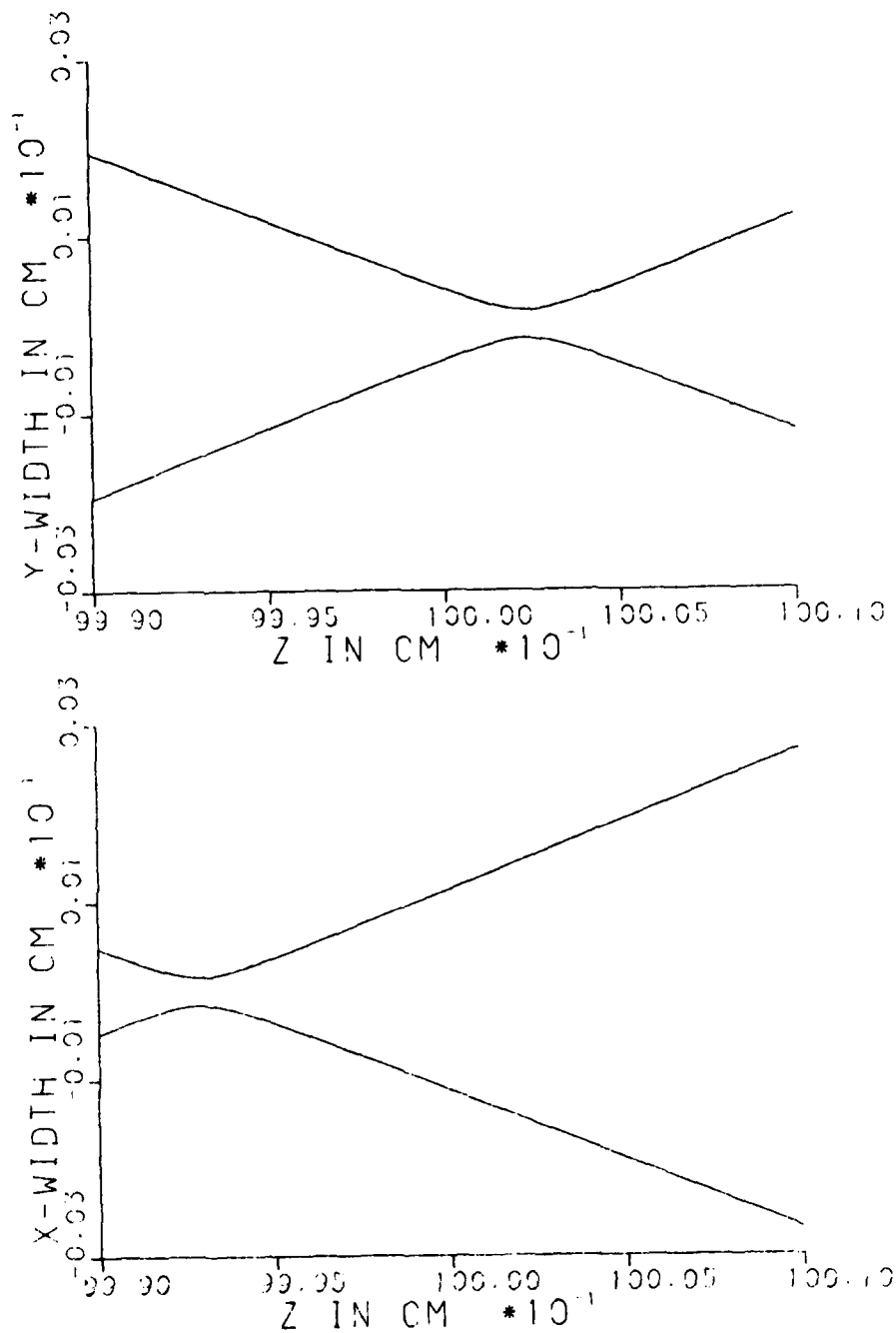


Figure 3.5. Propagation of beam width in the x and y direction for solution 4 of Table 3.3. $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$, $W_0 = \sqrt{2}$ cm, $f = 10$ cm and $\lambda = 6328\text{\AA}$.

IV. THE ROLE OF DIFFRACTION BY LENS APERTURE IN THE QUESTION OF UNIQUENESS. PART II: RADIUS OF BEAM \geq RADIUS OF LENS

In the previous section we have seen the nonuniqueness which arises when diffraction from the lens aperture plays no role in the propagation of the astigmatic Gaussian beam through a thin lens. In the present section we are going to look at what happens to that non-uniqueness as we decrease the size of the aperture such that diffraction from the aperture now plays an important role.

Figure 4.1 depicts the physical situation we are considering in this section. An astigmatic Gaussian beam is incident upon a thin lens of radius L surrounded by a circular aperture of variable radius, sL , where $0 < s \leq 1$.

The effective radius of the lens is then $s \cdot L$. The resultant focal plane intensities generated by incident fields which generated identical focal plane intensities for the case $W_0 \ll L$ will be studied for various values of s .

4.1 Calculation of $I_f(\vec{\rho})$

The incident field we will be using in this section is the same as that of Section III,

$$\begin{aligned}
 U_1(\vec{\rho}, 0) &= \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-\rho_0^2/W_0^2} e^{ika_3 2(\rho_0/L)^2 - 1} e^{ika_4 (\frac{x_0^2}{L^2} - \frac{y_0^2}{L^2})} e^{ika_5 2x_0 y_0 / L^2} \\
 &= \sqrt{\frac{2}{\pi}} \frac{e^{-ika_3 L^2/2}}{W_0} e^{-\rho_0^2/W_0^2} e^{ika_3 \rho_0^2} e^{ika_4 (x_0^2 - y_0^2)} e^{ika_5 2x_0 y_0},
 \end{aligned}
 \tag{4.1}$$

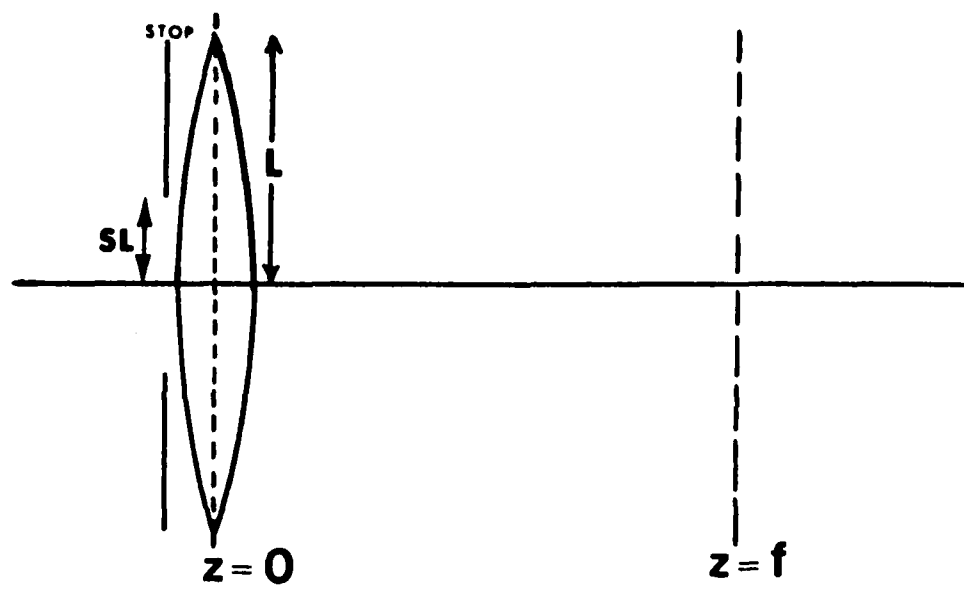


Figure 4.1. Thin Lens Geometry.

where

$$\alpha_3 = 2a_3/L^2, \quad (4.2)$$

$$\alpha_4 = a_4/L^2, \quad (4.3)$$

$$\alpha_5 = a_5/L^2. \quad (4.4)$$

The aperture plane field is therefore,

$$U_a(\vec{\rho}_0) = \sqrt{\frac{2}{\pi}} \frac{1}{W_0} e^{-ik\alpha_3 L^2/2} e^{-\rho_0^2/W_0^2} e^{ik\alpha_3 \rho_0^2} e^{ik\alpha_4(x_0^2 - y_0^2)} e^{ik\alpha_5 2x_0 y_0} \\ \times \text{circ}(\rho_0/sL). \quad (4.5)$$

Looking at Eq. (2.5) and (2.6) we see that the focal plane intensity is given by

$$I_f(\vec{\rho}) = |U_f(\vec{\rho})|^2, \\ = \frac{1}{\lambda^2 f^2} |\tilde{U}_a(\vec{\rho}/\lambda f)|^2, \quad (4.6)$$

where

$$\tilde{U}_a(\vec{\rho}/\lambda f) = \iint_{-\infty}^{\infty} U_a(\vec{\rho}_0) e^{-2\pi i(\vec{\rho}/\lambda f) \cdot \vec{\rho}_0} d^2 \rho_0. \quad (4.7)$$

Defining the spatial frequencies according to

$$\vec{\xi} = \vec{\rho}/\lambda f, \quad (4.8)$$

we see that in order to calculate $I_f(\vec{\rho})$,

$$\tilde{U}_a(\vec{\xi}) = \iint U_a(\vec{\rho}_0) e^{-2\pi i \vec{\xi} \cdot \vec{\rho}_0} d^2 \rho_0, \quad (4.9)$$

the two-dimensional spatial Fourier transform of $U_a(\vec{\rho}_0)$, must be evaluated. Since Eq. (4.9), with Eq. (4.5) substituted in, is not amenable to analytical evaluation we must evaluate it numerically.

The evaluation of $\tilde{U}_a(\vec{\xi})$, as given in Eq. (4.9), was performed by using a fast Fourier transform (FFT) algorithm to do the discrete Fourier transform (DFT) of $U_a(\vec{\rho}_0)$. The particular subroutine used was subroutine FFT developed by Posey¹⁹ in 1969 at Mississippi State University. This subroutine uses the FFT algorithm of Cooley and Tukey²⁰. For further details on the application of this subroutine to the problem we are considering, see reference 21.

4.2 Uniqueness in the Phase Retrieval Problem

We are now ready to look at the effect of the size of the aperture on the solutions of the previous section. Throughout this section we are going to look at astigmatic Gaussian beams of wavelength $\lambda = 6328\text{\AA}$, passing through a thin lens of radius 4 cm. and focal length 10 cm. The beamwidth in the aperture plane is taken to be $W_0 = \sqrt{2}$ cm.

Except for Figure 4.2 all the intensities computed in this section have as their input the function $U_a(\vec{\rho}_0)$ sampled at the points

$$\vec{\rho}_0 = (-8 \text{ cm.} + m\Delta, -8 \text{ cm.} + n\Delta) ,$$

where

$$m = 0, 1, 2, \dots, M-1$$

$$n = 0, 1, 2, \dots, N-1$$

and $\Delta = .125$ cm., $M = N = 128$. Also, except for Figure 4.2, all the focal plane intensities plotted out in this section correspond to the following focal plane points:

$$\vec{\rho} = (-2 \text{ cm}^{-1} + \frac{p}{M\Delta}, -2 \text{ cm}^{-1} + \frac{q}{N\Delta}) \lambda f ,$$

where M , N and Δ are as specified above and

$$p = 0, 1, 2, \dots, 64$$

$$q = 0, 1, 2, \dots, 64 .$$

4.2.1 Effect of Diffraction on a Solution with Circular Isophotes in the Focal Plane

In this subsection we compare the focal plane intensities generated by two fields which, according to Table 3.2, have different phases but produce the same focal plane intensities if $W_0 \ll L$. In particular we compare the focal plane intensities generated by

$$\text{Field A: } \alpha_4 = .75\lambda, \alpha_3 = \alpha_5 = 0$$

$$\text{Field B: } \beta_3 = .75\lambda, \beta_4 = \beta_5 = 0$$

for two different stop values.

Figures 4.2a and 4.2b show the focal plane intensities generated by Field A and Field B, respectively, for $s = 1$. In this case $W_0 = \sqrt{2}$ cm and $sL = 4$ cm, so W_0 is a good deal less than sL and the two fields create the same focal plane intensity patterns. (Note: In these two figures the input function was sampled at $M = N = 128$ and $\Delta = .0625$.

The focal plane intensities are plotted over the region $-4\lambda f \leq x \leq 4\lambda f$, $-4\lambda f \leq y \leq 4\lambda f$.)

Figures 4.3a and 4.3b show the focal plane intensities generated by Field A and Field B, respectively, when the stop is reduced to $\frac{1}{4}$. The two focal plane intensities are now quite different from each other. Indeed, Field A no longer creates a circularly symmetric intensity pattern.

Clearly, the nonuniqueness which was present when the beam was much smaller than the lens aperture is eliminated when the aperture plays a significant role. We have found this to be the case for each of several examples taken from Table 3.1 and 3.2 and expect it to be true in general except, of course, for the case where Field B is the twin of Field A. It will be shown in the next subsection, however, that in practice there exists a limit on how small the stop can be made in order to demonstrate this uniqueness.

4.4.2 Effect of Diffraction on a Solution with Elliptical Isophotes

In this subsection we compare the focal plane intensities generated by two fields, which, according to Table 3.3, have different phases but produce the same focal plane intensities if $W_0 \ll L$. In particular we compare the focal plane intensities generated by

$$\text{Field A: } \alpha_3 = .375\lambda, \alpha_4 = .1875\lambda, \alpha_5 = 0$$

$$\text{Field B: } \beta_3 = .1875\lambda, \beta_4 = .375\lambda, \beta_5 = 0$$

for several different stop values. Similar results hold true for cases where $\alpha_5 \neq 0$ and $\beta_5 \neq 0$, except in a rotated frame.

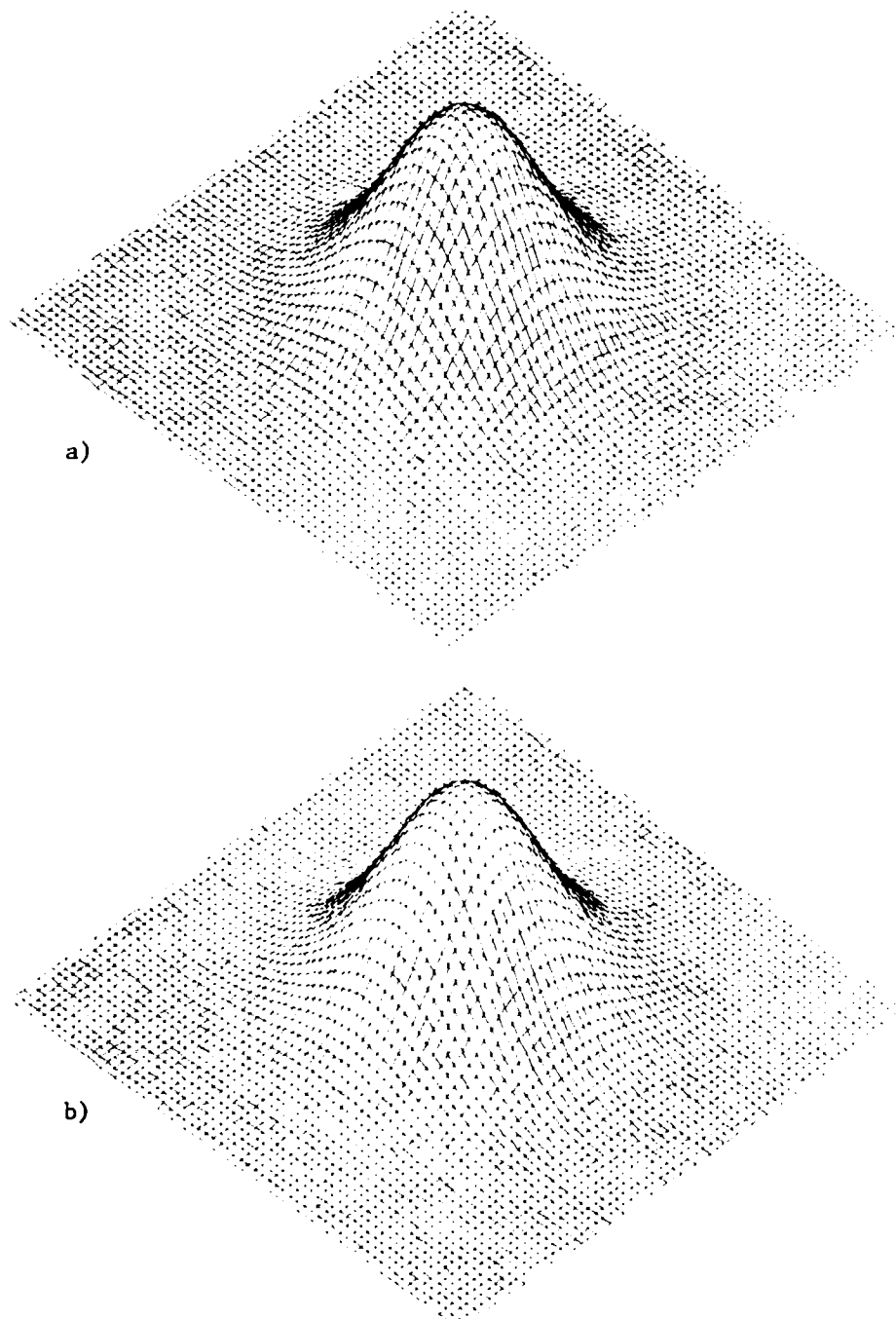


Figure 4.2. Focal plane intensity: stop = 1.

a) $\alpha_3 = 0, \alpha_4 = .75, \alpha_5 = 0$

b) $\beta_3 = .75\lambda, \beta_4 = 0, \beta_5 = 0$

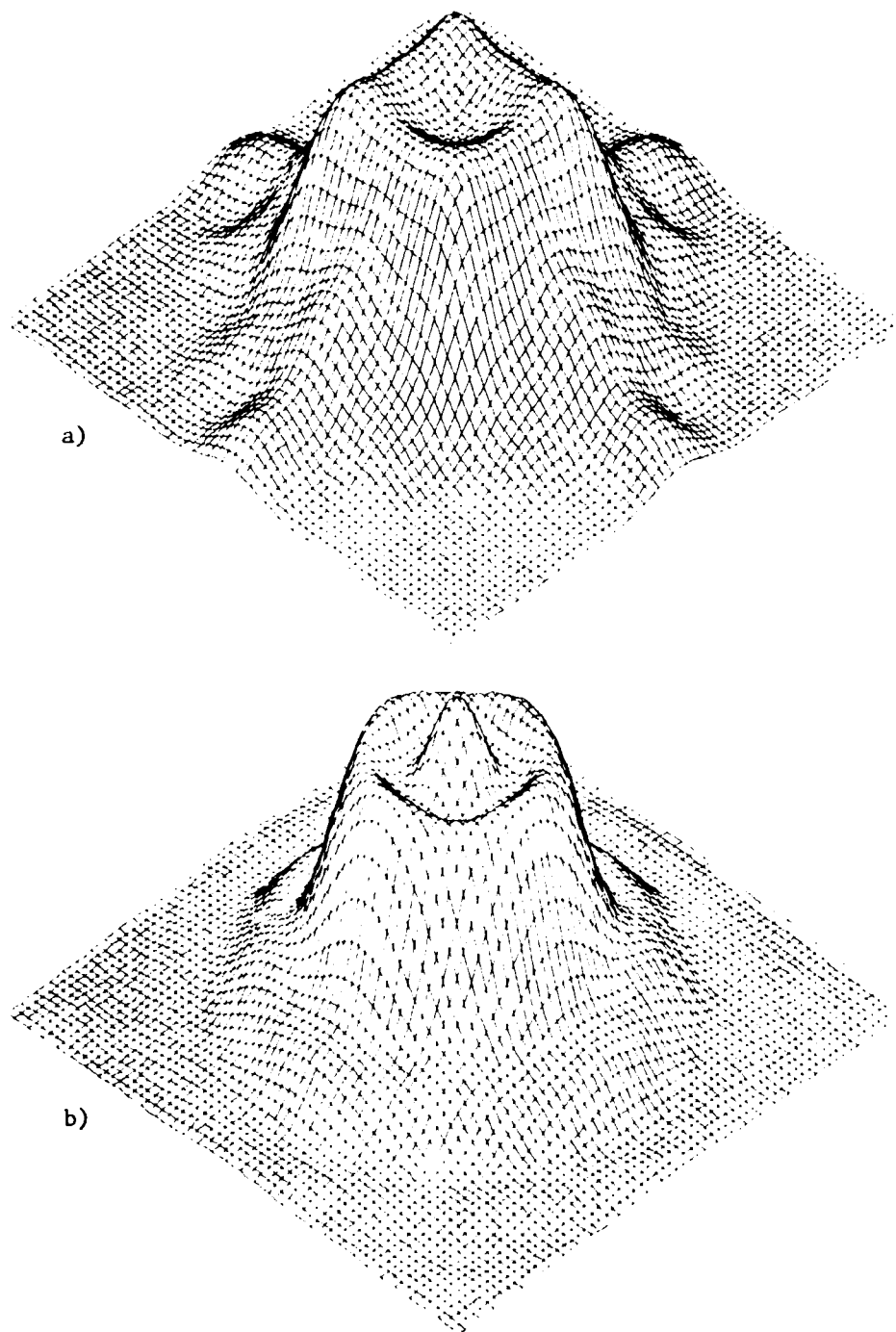


Figure 4.3. Focal plane intensity: stop = $\frac{1}{4}$.

a) $\alpha_3 = 0, \alpha_4 = .75\lambda, \alpha_5 = 0$

b) $\beta_3 = .75\lambda, \beta_4 = 0, \beta_5 = 0$

Figures 4.4a and 4.4b show the focal plane intensities generated by Field A and Field B, respectively, for $s = 1$. In this case $W_0 = \sqrt{2}$ cm and $sL = 4$ cm, so W_0 is a good deal less than sL and the two fields produce the same focal plane intensities.

Figures 4.5, 4.6 and 4.7 each compare the focal plane intensities generated by the two fields. The stop values for the three sets of figures are $s = 3/8, \frac{1}{2}, \frac{1}{4}$ respectively. Clearly, due to the diffraction from the aperture, Field A and Field B no longer produce the same intensity pattern and the nonuniqueness which was evident in Figure 4.4 has been eliminated.

In Figure 4.8 the stop has been decreased even further, to $s = 1/8$. Here the two fields again generate the same focal plane intensity patterns, so there is some limit upon how small we can make the stop and still demonstrate the uniqueness. Physically, what is happening in this case is that the aberration and the stop are so small that the aperture plane fields for Field A and Field B are almost identical.

It should be noted at this point that decreasing the sign of the stop will never allow us to distinguish between two fields which are twins of each other, e.g., the first and second solution in Table 3 or the third and fourth solution of that table. More will be said in the next section about the twin solution and a method for eliminating it will be suggested.

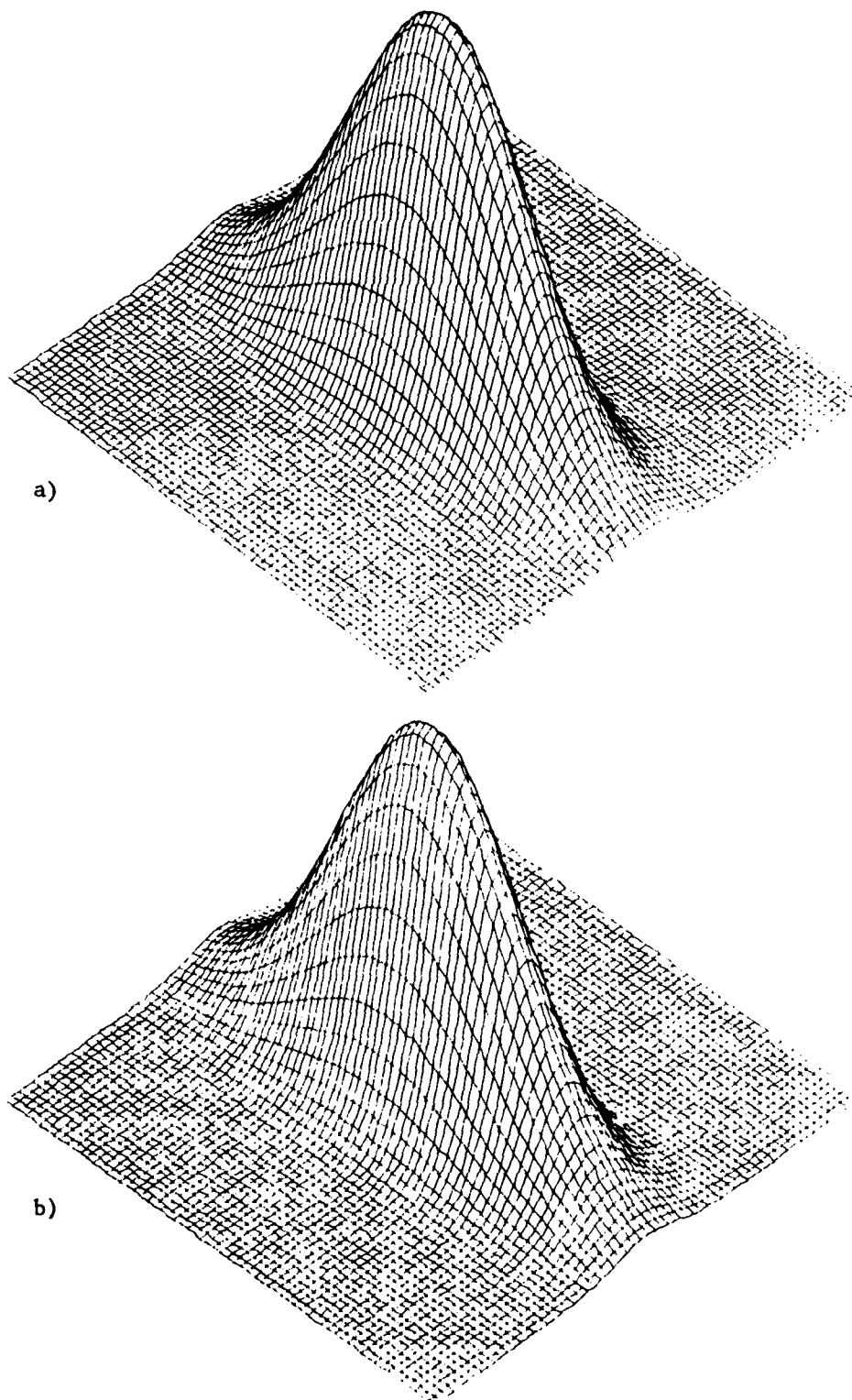


Figure 4.4. Focal plane intensity: stop = 1.
 a) $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$
 b) $\beta_3 = .1875\lambda$, $\beta_4 = .375\lambda$, $\beta_5 = 0$

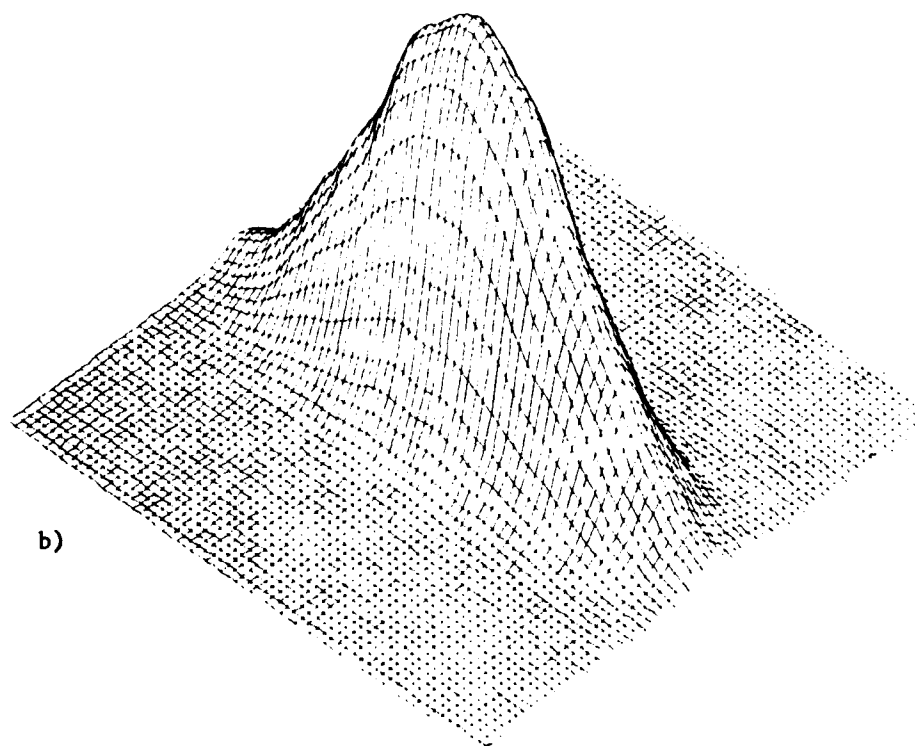
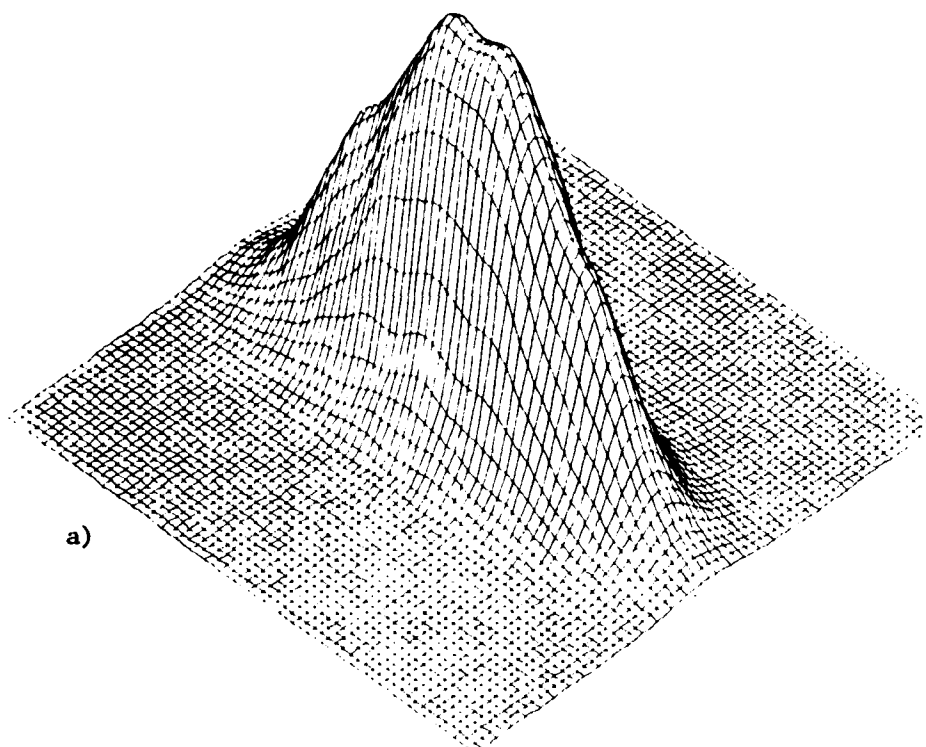
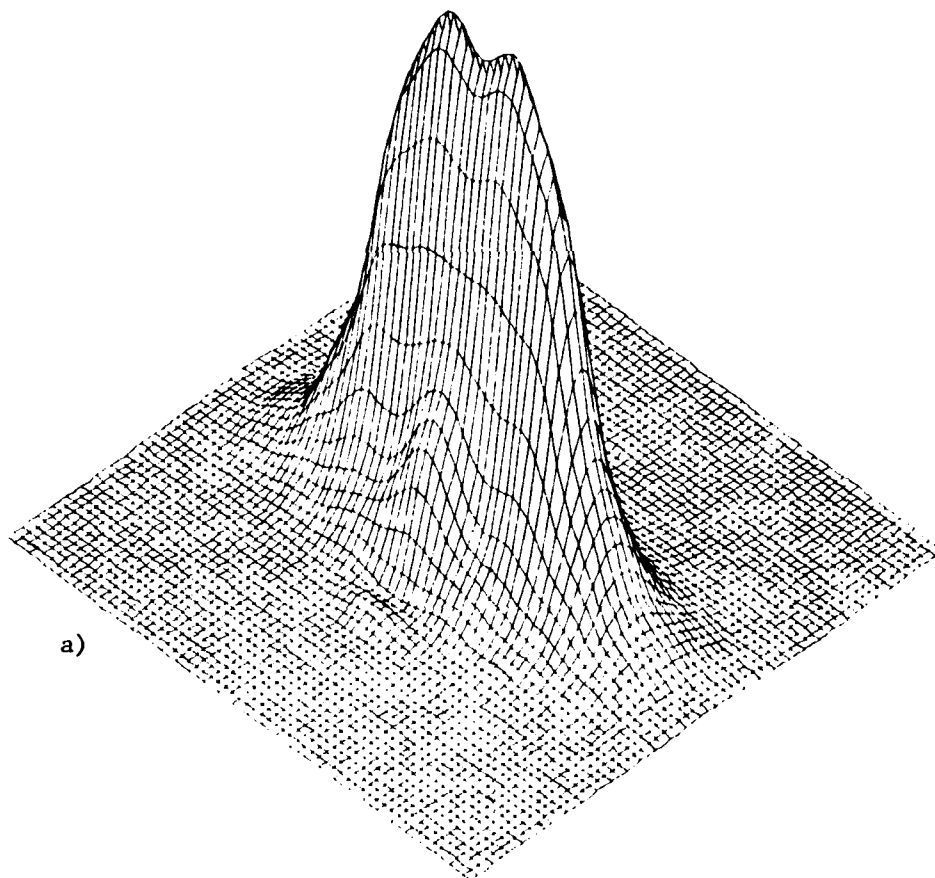
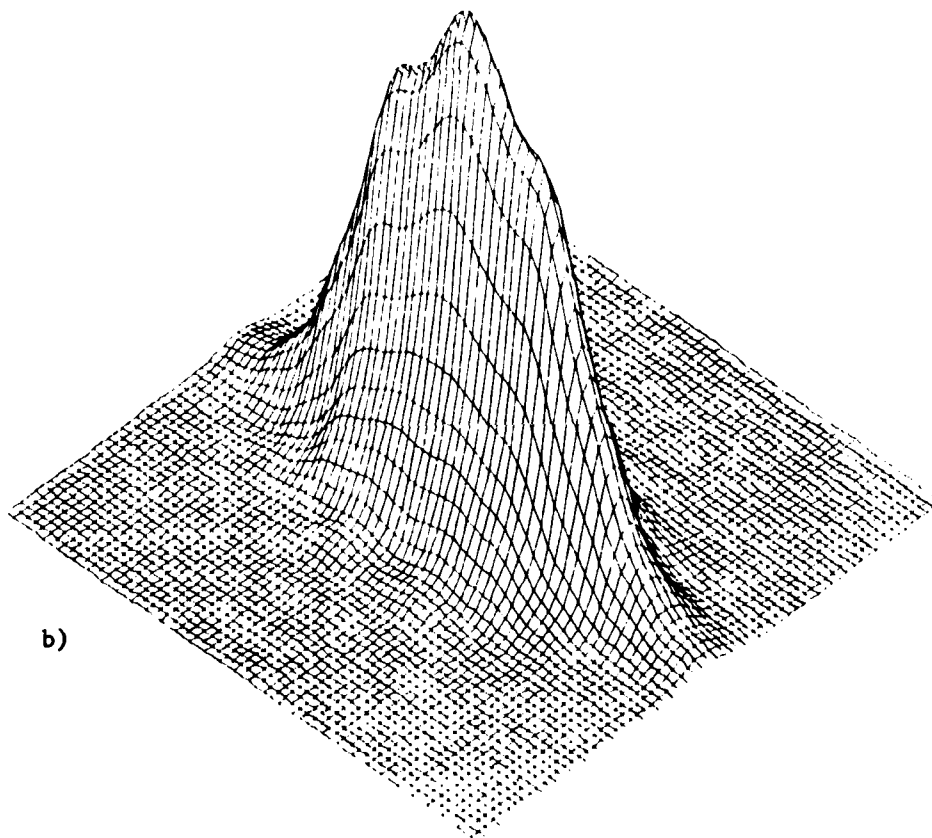


Figure 4.5. Focal plane intensity: stop = $\frac{1}{2}$.
a) $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$
b) $\beta_3 = .1875\lambda$, $\beta_4 = .375\lambda$, $\beta_5 = 0$



a)

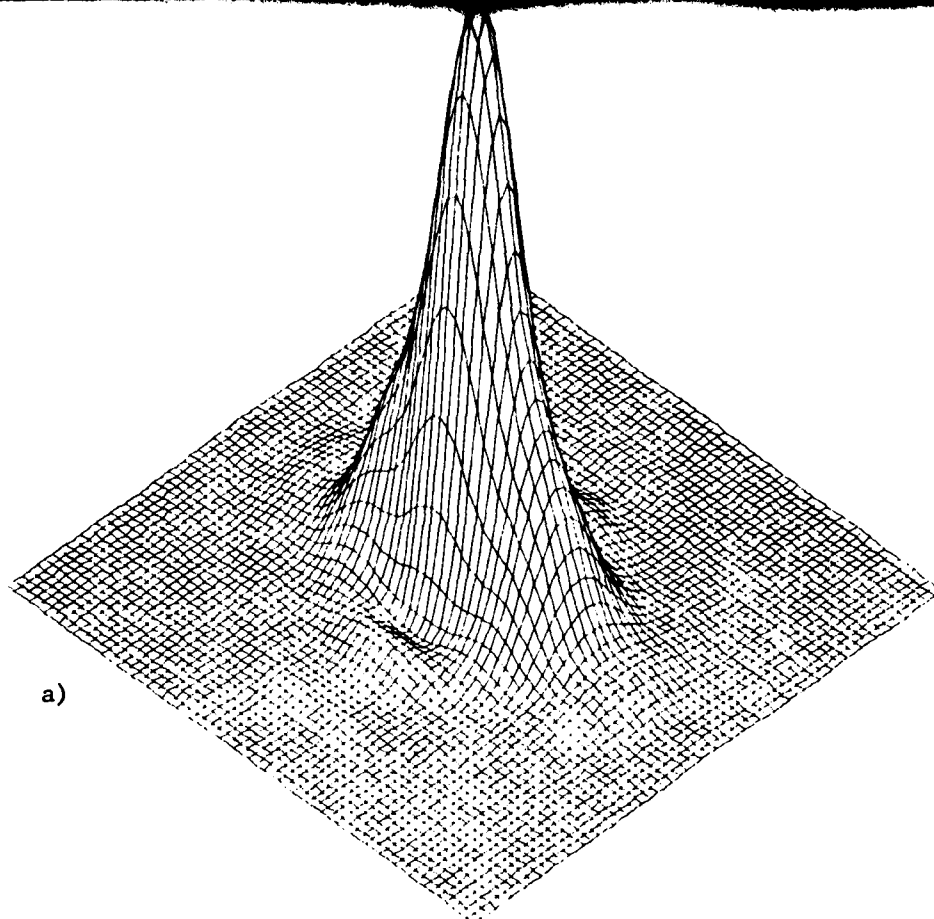


b)

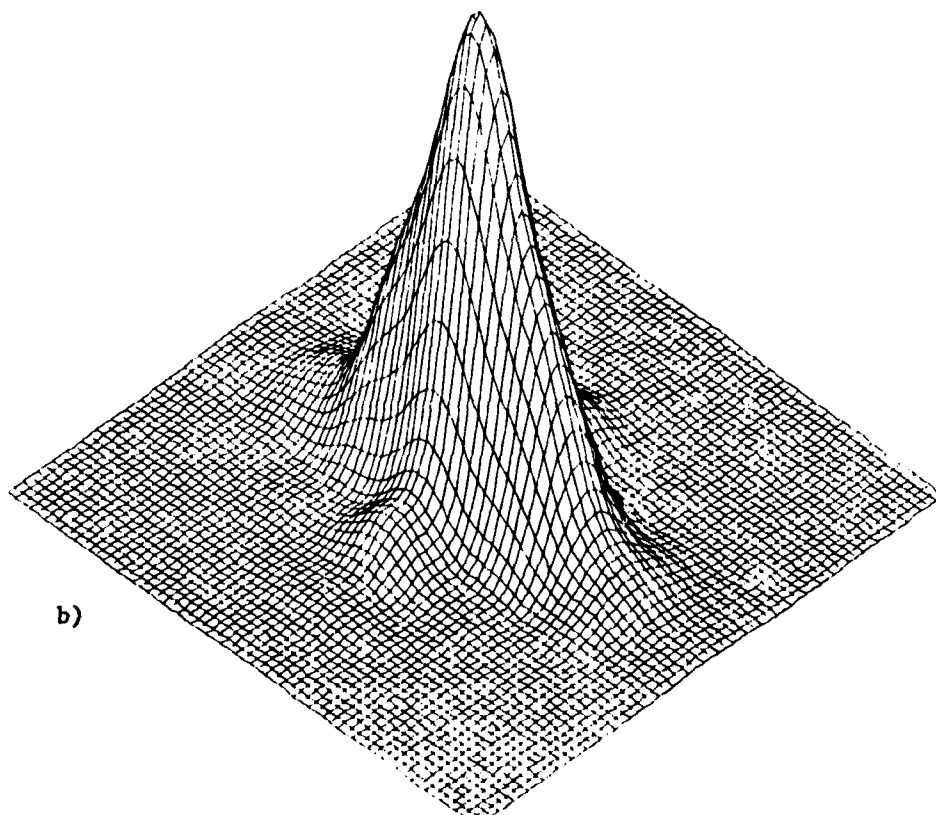
Figure 4.6. Focal plane intensity: stop = $3/8$.

a) $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$

b) $\beta_3 = .1875\lambda$, $\beta_4 = .375\lambda$, $\beta_5 = 0$



a)



b)

Figure 4.7. Focal plane intensity: stop = $\frac{1}{4}$.
 a) $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$
 b) $\beta_3 = .1875\lambda$, $\beta_4 = .375\lambda$, $\beta_5 = 0$

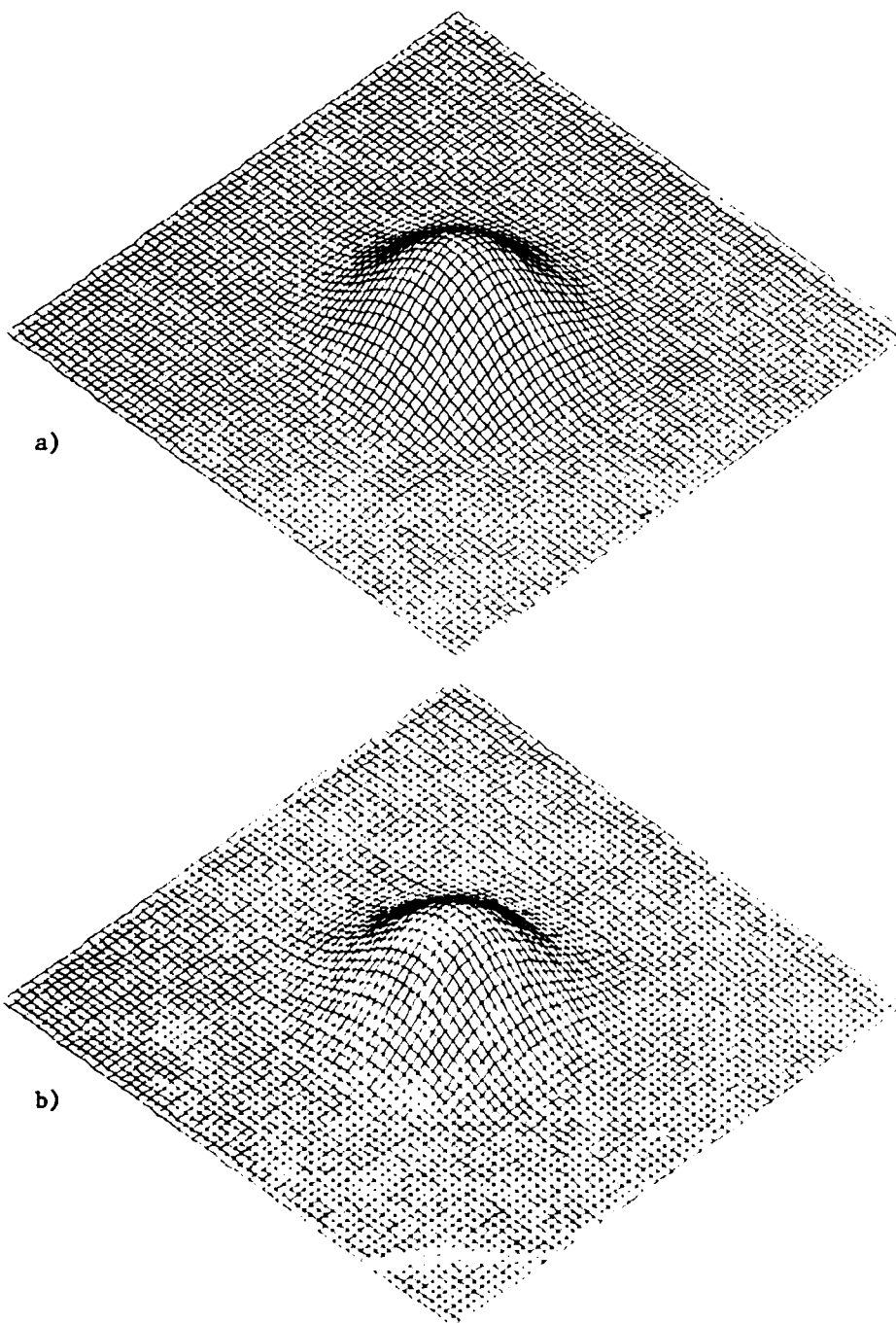


Figure 4.8. Focal plane intensity: stop = 1/8.
 a) $\alpha_3 = .375\lambda$, $\alpha_4 = .1875\lambda$, $\alpha_5 = 0$
 b) $\beta_3 = .1875\lambda$, $\beta_4 = .375\lambda$, $\beta_5 = 0$

V. SUMMARY AND TWIN SOLUTION

In this section the "twin solution" ambiguity will be discussed and a method to eliminate it will be proposed. As we saw in Section III, the twin solution is always possible for the types of aperture plane fields we have used so far. In fact, the twin solution is always possible for any aperture plane field that has inversion symmetry.

5.1 Summary and Twin Solution

Phase retrieval problems for cases where the incident field is in the form of an astigmatic Gaussian beam have been studied. It was found that for cases where the radius of the beam is much smaller than the radius of the lens, phase retrieval is nonunique.

More precisely, for cases where the aperture plane and focal plane intensities are circularly symmetric, there exists an infinite number of solutions to the phase retrieval problem if the focal plane intensities have circular isophotes. If they have elliptical isophotes, then there are only four possible solutions to the problem, two distinctly different solutions, each with its own "twin".

For the numerical examples we did, letting the lens be surrounded by a circular aperture of significant radius (in comparison with the beam radius) will then reduce the nonuniqueness to only the twin solution.

We shall now discuss the twin solution further. Let us now consider the incident field

$$U_{1A}(\vec{\rho}_0, 0) = |U_1(\vec{\rho}_0, 0)| e^{ikW(\vec{\rho}_0)} \quad (5.1)$$

and its twin

$$U_{iB}(\vec{\rho}_0, 0) = |U_i(\vec{\rho}_0, 0)| e^{-ikW(-\vec{\rho}_0)} . \quad (5.2)$$

The corresponding aperture plane fields are

$$U_{aA}(\vec{\rho}_0) = U_{iA}(\vec{\rho}_0, 0) P(\vec{\rho}_0) , \quad (5.3)$$

$$U_{aB}(\vec{\rho}_0) = U_{iB}(\vec{\rho}_0, 0) P(\vec{\rho}_0) , \quad (5.4)$$

where $P(\vec{\rho}_0)$ represents the pupil function.

It follows from Eqs. (5.1) - (5.4) that

$$U_{aA}(\vec{\rho}_0) = |U_a(\vec{\rho}_0)| e^{ikW(\vec{\rho}_0)} , \quad (5.5)$$

$$U_{aB}(\vec{\rho}_0) = |U_a(\vec{\rho}_0)| e^{-ikW(-\vec{\rho}_0)} , \quad (5.6)$$

where

$$U_a(\vec{\rho}_0) = |U_i(\vec{\rho}_0, 0)| P(\vec{\rho}_0) . \quad (5.7)$$

Theorem

If

$$|U_a(-\vec{\rho}_0)| = |U_a(\vec{\rho}_0)| , \text{ for all } \vec{\rho}_0 , \quad (5.8)$$

then

$$\hat{U}_{aB}(\vec{\xi}) = [\hat{U}_{aA}(\vec{\xi})]^* , \text{ for all } \vec{\xi} . \quad (5.9)$$

Proof:

It follows from Eqs. (5.5) and (5.6) and the definition of the Fourier transform that

$$\tilde{U}_{aA}(\xi) = \iint_{\infty} |U_a(\vec{\rho}_0)| e^{ikW(\vec{\rho}_0)} e^{-2\pi i \vec{\xi} \cdot \vec{\rho}_0} d^2 \rho_0, \quad (5.10)$$

$$\tilde{U}_{aB}(\xi) = \iint_{\infty} |U_a(\vec{\rho}_0)| e^{-ikW(-\vec{\rho}_0)} e^{-2\pi i \vec{\xi} \cdot \vec{\rho}_0} d^2 \rho_0. \quad (5.11)$$

If we define $\vec{\eta} = -\vec{\rho}_0$, Eq. (5.11) can be rewritten as

$$\tilde{U}_{aB}(\xi) = \iint_{\infty} |U_a(-\vec{\eta})| e^{-ikW(\vec{\eta})} e^{2\pi i \vec{\xi} \cdot \vec{\eta}} d^2 \eta. \quad (5.12)$$

Using Eq. (5.8) in Eq. (5.12) yields

$$\begin{aligned} \tilde{U}_{aB}(\vec{\eta}) &= \iint_{\infty} |U_a(\vec{\eta})| e^{-ikW(\eta)} e^{2\pi i \vec{\xi} \cdot \vec{\eta}} d^2 \eta, \\ &= \left[\tilde{U}_{aA}(\vec{\xi}) \right]^*. \end{aligned} \quad (5.13)$$

It follows from Eq. (2.7) that

$$I_{fB}(\vec{\rho}) = I_{fA}(\vec{\rho}) \quad \text{for all } \vec{\rho}. \quad (5.14)$$

From Eqs. (5.7) and (5.8) we see that if

$$|U_1(-\vec{\rho}_0, 0)| = |U_1(\vec{\rho}_0, 0)|, \quad (5.15)$$

and

$$P(-\vec{\rho}_0) = P(\vec{\rho}_0), \quad (5.16)$$

then $|U_a(-\vec{\rho}_0)| = |U_a(\vec{\rho}_0)|$ and the twin solution will occur.

Note that all the incident fields discussed so far have circularly symmetric intensities and thus satisfy Eq. (5.15). Also, the circular aperture used satisfies Eq. (5.16). As a result, twin solution is always a possibility in Section III and IV.

If the pupil function is not of the type in Eq. (5.16), then the

theorem will not be applicable. Therefore, if we have a nonsymmetric pupil, the twin solution might not occur. In the next section we will look at one particular example where this is true.

5.2 An Example with Nonsymmetric Pupil

A numerical example of the same nature as in Section IV was performed to determine the effect of a triangular aperture on the twin solution of the incident astigmatic Gaussian beam with $\alpha_3 = 0$, $\alpha_4 = .75\lambda$ and $\alpha_5 = 0$, $\lambda = 6328\text{\AA}$ and $W_0 = \sqrt{2}$ cm. The aperture used was an equilateral triangle inscribed into the lens aperture for a lens of radius 1 cm and focal length 10 cm.

The focal plane intensity for the corresponding aperture plane field is computed in the same manner as in Subsection 4.4.2 and plotted out in Figure 5.1a together with the plot for its "twin" in Figure 5.1b. It is evident from the plot that for this particular case, the use of a nonsymmetric aperture eliminates the "twin solution" ambiguity. More studies must be made before any general statement can be made about the use of nonsymmetric aperture to eliminate the "twin solution".

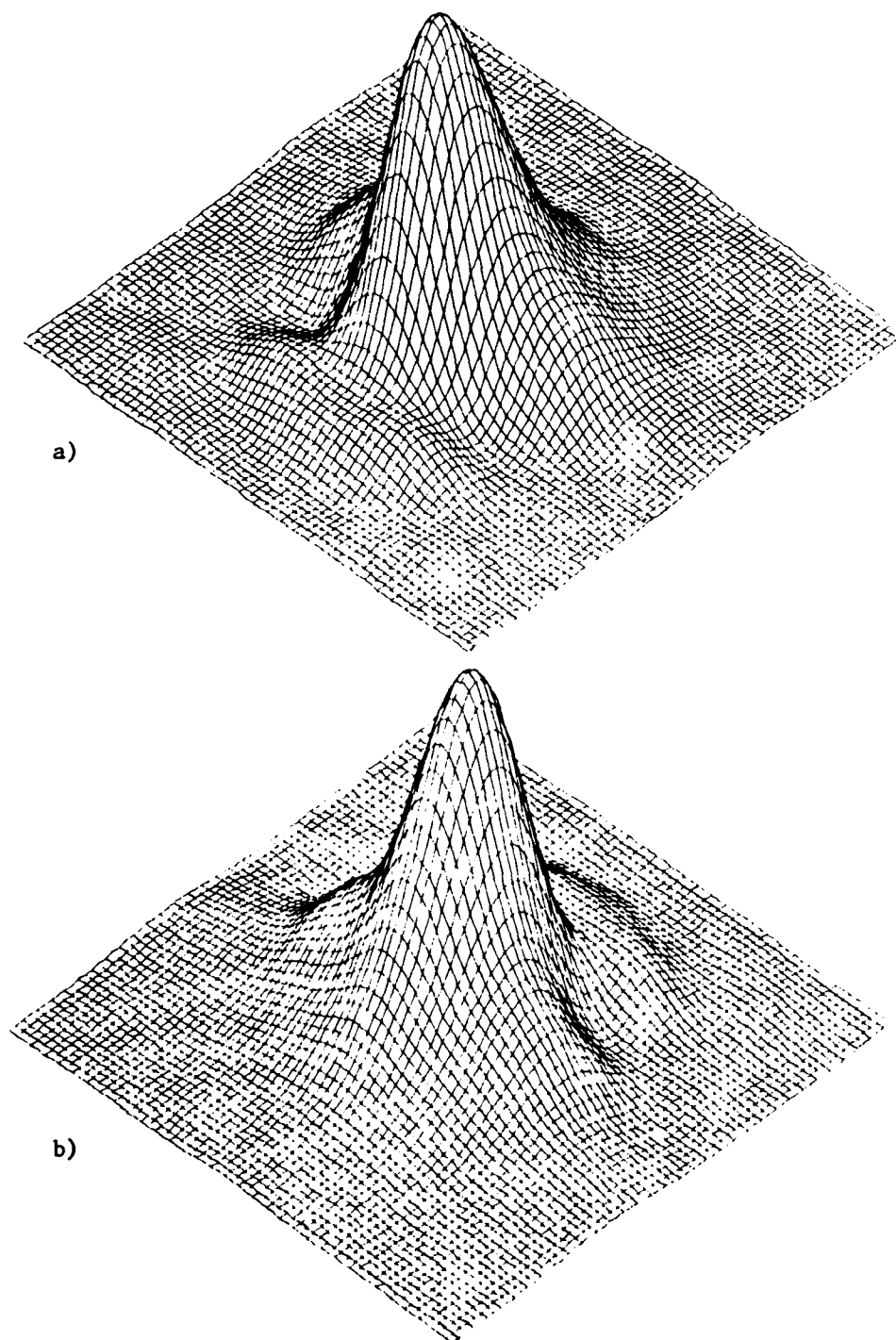


Figure 5.1. Focal plane intensity, triangular aperture.

a) $\alpha_3 = 0, \alpha_4 = .75\lambda, \alpha_5 = 0$
b) $\beta_3 = 0, \beta_4 = -.75\lambda, \beta_5 = 0$

APPENDIX A

CIRCULAR ISOPHOTES IN THE FOCAL PLANE FOR SOLUTIONS IN TABLE 3.1 AND

TABLE 3.2

The intensities in the focal plane for an astigmatic Gaussian beam incident on a thin lens is given by Eq. (3.42) with $z = f$,

$$\bar{I}(\bar{x}, \bar{y}, z=f) = \frac{2}{\pi} \frac{e^{-Z\bar{x}^2/W_x^2(f)}}{W_x(f)} \frac{e^{-Z\bar{y}^2/W_y^2(f)}}{W_y(f)}, \quad (\text{A.1})$$

where

$$W_x(f) = W_0 \sqrt{\left(1 + \frac{f}{R_{x_0}}\right)^2 + \left(\frac{f}{z_R}\right)^2}, \quad (\text{A.2a})$$

$$W_y(f) = W_0 \sqrt{\left(1 + \frac{f}{R_{y_0}}\right)^2 + \left(\frac{f}{z_R}\right)^2}, \quad (\text{A.2b})$$

Looking at Eq. (A.1) we see that $I_f(\vec{\rho})$ is circularly symmetric if and only if

$$W_x(f) = W_y(f). \quad (\text{A.3})$$

Substituting Eq. (3.36) into Eq. (A.2) we then have

$$W_x(f) = W_0 f \sqrt{4(\alpha_3 + \bar{\alpha}_4)^2 + \left(\frac{1}{z_R}\right)^2}, \quad (\text{A.4})$$

$$W_y(f) = W_0 f \sqrt{4(\alpha_3 - \bar{\alpha}_4)^2 + \left(\frac{1}{z_R}\right)^2}. \quad (\text{A.5})$$

Thus, for cases with $\bar{a}_4 = 0$ (Table 3.1), or $a_3 = 0$ (Table 3.2) it follows that

$$W_x(f) = W_y(f) \quad .$$

Hence, for these cases, the focal plane intensities are circularly symmetric.

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